

ALGORITHMIC ASPECTS OF BRANCHED COVERINGS II/V. SPHERE BISETS AND THEIR DECOMPOSITIONS

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ABSTRACT. We consider the action of mapping class groups, by pre- and post-composition, on branched coverings, and encode them algebraically as *mapping class bisets*. We show how the mapping class biset of maps preserving a multi-curve decomposes into mapping class bisets of smaller complexity, called *small mapping class bisets*.

We phrase the decision problem of *Thurston equivalence* between branched self-coverings of the sphere in terms of the conjugacy and centralizer problems in a mapping class biset, and use our decomposition results on mapping class bisets to reduce these decision problems to small mapping class bisets.

This is the main step in our proof of decidability of Thurston equivalence, since decidability of conjugacy and centralizer problems in the small mapping class bisets are well understood in terms of linear algebra, group theory and complex analysis.

Branched coverings themselves are also encoded into bisets, with actions of the fundamental groups. We characterize those bisets that arise from branched coverings between topological spheres, and extend this correspondence to maps between spheres with multicurves, whose algebraic counterparts are *sphere trees of bisets*.

A concrete outcome of our investigations is the construction of a Thurston map with infinitely generated centralizer — while centralizers of homeomorphisms are always finitely generated.

1. INTRODUCTION

Consider a topological sphere S^2 and a finite subset $A \subset S^2$. A *Thurston map* is a branched covering $f: (S^2, A) \hookrightarrow (S^2, A)$ such that A contains the critical values of f . It is natural to consider Thurston maps up to *isotopy*: $f_0, f_1: (S^2, A) \hookrightarrow (S^2, A)$ are isotopic, written $f_0 \approx f_1$, if there is a path of Thurston maps $(f_t: (S^2, A) \hookrightarrow (S^2, A))_{t \in [0,1]}$ connecting them. We denote by $K(S^2, A)$ the semigroup of isotopy classes of Thurston maps $(S^2, A) \hookrightarrow (S^2, A)$.

In particular, those isotopy classes of Thurston maps that fix A and are homeomorphisms form a group $\mathbf{Mod}(S^2, A)$, the *pure mapping class group*. Two Thurston maps $f: (S^2, A) \hookrightarrow (S^2, A)$ and $g: (S^2, C) \hookrightarrow (S^2, C)$ are *combinatorially equivalent*, written $f \sim g$, if there exists a homeomorphism $\phi: (S^2, A) \rightarrow (S^2, C)$ with $\phi \circ f \approx g \circ \phi$. Equivalently, $f_0 \sim f_1$ if there exists a path of Thurston maps $(f_t: (S^2, A_t) \hookrightarrow (S^2, A_t))_{t \in [0,1]}$ connecting them, see Definition 2.11 and Lemma 2.12.

For homeomorphisms, combinatorial equivalence amounts to conjugacy, and was proven to be decidable by Hemion [18]. In this article (the second of a series of five, see [2] for an overview) we consider general decision problems for Thurston maps:

Date: October 7, 2016.

Partially supported by ANR grant ANR-14-ACHN-0018-01 and DFG grant BA4197/6-1.

The conjugacy problem: Given $f, g \in K(S^2, A)$, are they conjugate? If so, give a witness $m \in \mathbf{Mod}(S^2, A)$ to $mg = fm$.

The centralizer problem: Given $f \in K(S^2, A)$, compute its centralizer $Z(f) := \{m \in \mathbf{Mod}(S^2, A) \mid mf = fm\}$.

To determine whether two Thurston maps $f: (S^2, A) \hookrightarrow (S^2, A)$ and $g: (S^2, C) \hookrightarrow (S^2, C)$ are combinatorially equivalent, enumerate all bijections $\phi: A \rightarrow C$, and extend each ϕ arbitrarily to a homeomorphism $\hat{\phi}: (S^2, A) \rightarrow (S^2, C)$. Then $f \sim g$ if and only if f and $\hat{\phi}^{-1} \circ g \circ \hat{\phi}$ are conjugate in $K(S^2, A)$ for some $\phi: A \rightarrow C$.

A *multicurve* on (S^2, A) is a collection \mathcal{C} of disjoint pairwise non-homotopic simple closed curves on $S^2 \setminus A$, none of which can be homotoped rel A to a point, and viewed up to isotopy. We let $K(S^2, A, \mathcal{C})$ denote those Thurston maps $f \in K(S^2, A)$ that preserve \mathcal{C} in the sense that $f^{-1}(\mathcal{C}) \supseteq \mathcal{C}$, and we denote by $\mathbf{Mod}(S^2, A, \mathcal{C})$ the subgroup of $\mathbf{Mod}(S^2, A)$ consisting of mapping classes that preserve \mathcal{C} curvewise, including their orientation.

Consider (S^2, A) and a multicurve \mathcal{C} . The connected components of $S^2 \setminus \mathcal{C}$ are called *small spheres*, and are themselves homeomorphic to punctured spheres. Given $f \in K(S^2, A, \mathcal{C})$ and a periodic component $S \subset S^2 \setminus \mathcal{C}$ viewed as a punctured sphere \hat{S} , the first return map $f^e: \hat{S} \hookrightarrow \hat{S}$ is called a *small Thurston map*. We denote by $R(f, A, \mathcal{C})$ the set of small Thurston maps of f .

The main result of this article is the following theorem; briefly said, it reduces the decision problems for a Thurston map to that of its small Thurston maps:

Theorem A (See Theorem 6.21). *Let (S^2, A) be a punctured sphere, and let \mathcal{C} be a multicurve on (S^2, A) . There is then an algorithm with oracle that,*

- *given two Thurston maps $f, g \in K(S^2, A, \mathcal{C})$,*
- *assuming that the centralizers of all small Thurston maps in $R(f, A, \mathcal{C})$ are computable (e.g. as finite-index subgroups of products of mapping class groups),*
- *assuming that the conjugacy problems between maps in $R(f, A, \mathcal{C})$ and in $R(g, A, \mathcal{C})$ are solvable, and that witnesses can be produced in case maps are conjugate,*

answers whether f, g are conjugate under $\mathbf{Mod}(S^2, A, \mathcal{C})$, if so produces a witness, and computes the centralizer of f as the kernel of a homomorphism from a finite-index subgroup of a product of mapping class groups towards a finitely generated abelian group.

We show by an example (see §8.3) that this description of centralizers of Thurston maps is in a sense the best that can be achieved; this makes centralizers of Thurston maps significantly more complicated than centralizers of mapping classes, which are always finite-index subgroups of products of mapping class groups.

1.1. Decidability of combinatorial equivalence. As an important outcome of Nielsen-Thurston classifications, centralizers of homeomorphisms are computable as finite-index subgroups of products of mapping class groups. Thurston's rigidity theorem implies, on the other hand, that rational maps with hyperbolic orbifold have trivial centralizer. Centralizers of affine maps on the torus are computable as linear groups.

By [selinger-yampolsky:geometrization], for every Thurston map f there exists a computable, canonically defined multicurve \mathcal{C}_f (Pilgrim's "canonical obstruction") such that all small Thurston maps are of the above type. Therefore, to

compute the centralizer of f , it suffices to know how to combine the centralizers of these small Thurston maps and Dehn twists along \mathcal{C}_f , and this is what Theorem A does.

We note that the computation of centralizers is essential to Theorem A; it is by solving equations between centralizers of small Thurston maps that the global conjugacy and centralizer problems are solved.

We remark that, philosophically, it is enough to compute centralizers to determine combinatorial equivalence; indeed to determine whether two Thurston maps f, g are combinatorially equivalent, combine them into a self-map $h = f \sqcup g$ on the disjoint union of two spheres; then $f \sim g$ if and only if the centralizer of h contains an element exchanging both spheres. We chose not to make explicit use of this trick, which would require us to extend all results to disjoint unions of spheres.

We refer to [2] for more details on the overall strategy, algorithms, and examples of Thurston maps and combinatorial equivalence.

1.2. Algebraic structure of branched coverings. In this article, we specialize results of [3] to 2-dimensional spheres and Thurston maps. It is in fact worthwhile to consider a slightly more general situation: fix a branched covering $f: (S^2, C) \rightarrow (S^2, A)$ such that A contains $f(C)$ and the critical values of f ; we call f a *sphere map*. Set

$$(1) \quad M(f, C, A) = \{m' f m'' \mid m' \in \mathbf{Mod}(S^2, C), m'' \in \mathbf{Mod}(S^2, A)\} / \text{isotopy},$$

the composition being written in algebraic (left-to-right) order. Then $M(f, C, A)$ is a *biset*: a set endowed with commuting left and right actions by $\mathbf{Mod}(S^2, C)$ and $\mathbf{Mod}(S^2, A)$ respectively. We call it the *mapping class biset* of f . If $f: (S^2, A) \hookrightarrow (S^2, A)$ is a Thurston map, $M(f, A, A)$ is a subbiset of $K(S^2, A)$.

We give an algorithm (Algorithm 6.11) that computes the structure of $M(f, C, A)$ as a biset. It uses in a fundamental manner the facts that $H := \pi_1(S^2 \setminus C)$ and $G := \pi_1(S^2 \setminus A)$ are free groups, and that $\mathbf{Mod}(S^2, C)$ and $\mathbf{Mod}(S^2, A)$ act faithfully respectively on H and G by outer automorphisms. Decompositions of (certain variants of) mapping class bisets are important steps in proving Theorem A; see for example §6.4 and §6.5.

Sphere maps $f: (S^2, C) \rightarrow (S^2, A)$ themselves are represented as H - G -bisets $B(f)$, see (4) and [3, §5]. It was shown by Kameyama [20] that $B(f)$ is a complete invariant of isotopy, namely $B(f) \cong B(g)$ if and only if f, g are isotopic. We give a converse to this result in Theorem 2.8 by showing that to every sphere biset there is an associated sphere map, unique up to isotopy. This allows us to switch freely between geometric and algebraic settings of sphere maps and bisets.

Recall from [3] that left-free bisets are naturally associated with topological correspondences, namely pairs of maps $Y \leftarrow Z \rightarrow X$ such that $Z \rightarrow X$ is a covering. The mapping class biset $M(f, C, A)$ is the biset of a well-known correspondence, one between the moduli spaces \mathcal{M}_C and \mathcal{M}_A ; see Proposition 8.1 in §8.

Multicurves \mathcal{C} on (S^2, A) are represented as a collection of conjugacy classes in G . The main result from [3] is that, given a 1-dimensional cover of a space such as that afforded by the small spheres of (S^2, A) , and a map compatible with this cover such as a Thurston map f preserving \mathcal{C} , there exists a tree of bisets decomposition of f whose “fundamental biset” is isomorphic to $B(f)$. We show that this decomposition is computable:

Theorem B (See Theorem 3.9). *There is an algorithm that, given a Thurston map $f: (S^2, A) \hookrightarrow$ by its biset $B(f)$ and given a multicurve \mathcal{C} on (S^2, A) with $f^{-1}(\mathcal{C}) \supseteq \mathcal{C}$, computes the tree of bisets decomposition of $B(f)$ along \mathcal{C} .*

We extend Kameyama's result to maps with multicurves: we define *sphere trees* of bisets in Definition 4.12, and prove that the decomposition of a sphere map is a sphere tree of bisets and conversely, giving a complete invariant up to isotopy or combinatorial equivalence:

Theorem C (See Theorem 3.10, Corollary 3.11 and Corollary 4.15). *Let $f, f': (S^2, A, \mathcal{C}) \hookrightarrow$ be Thurston maps with the same (possibly empty) multicurve \mathcal{C} satisfying $f^{-1}(\mathcal{C}) \supseteq \mathcal{C} \subseteq (f')^{-1}(\mathcal{C})$. Then f, f' are isotopic rel $A \cup \mathcal{C}$ if and only if the sphere trees of bisets of f, f' are isomorphic.*

Consequently, if $g: (S^2, C, \mathcal{D}) \hookrightarrow$ is a Thurston map with \mathcal{D} satisfying $g^{-1}(\mathcal{D}) \supseteq \mathcal{D}$, then f and g are combinatorially equivalent by a homeomorphism sending A to C and \mathcal{C} to \mathcal{D} if and only if the sphere trees of bisets of f, g are conjugate.

Finally, for every sphere tree of bisets $\mathfrak{Y}\mathfrak{B}_{\mathfrak{X}}$ with spheres (S^2, C, \mathcal{D}) and (S^2, A, \mathcal{C}) associated respectively with the trees of groups \mathfrak{Y} and \mathfrak{X} there exists a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$, unique up to isotopy rel $C \cup \mathcal{D}$, whose graph of bisets is isomorphic to \mathfrak{B} .

Theorem C expresses, in the algebraic language of bisets, the decomposition and combination theorems of Kevin Pilgrim (see e.g. [29, Theorem 5.1]). Numerous examples of sphere tree of bisets decompositions appeared in [2, §7]. Here two graphs of bisets $\mathfrak{X}\mathfrak{B}_{\mathfrak{X}}$ and $\mathfrak{Y}\mathfrak{C}_{\mathfrak{Y}}$ are called conjugate if there exists a biprincipal (see [3, §3.8]) tree of bisets $\mathfrak{X}\mathfrak{I}_{\mathfrak{Y}}$ such that $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{I}$ and $\mathfrak{I} \otimes_{\mathfrak{Y}} \mathfrak{C}$ are isomorphic, namely have same underlying graphs and isomorphic bisets and intertwiners.

We extend in §7 the equivalence between sphere maps and bisets to the setting of *orbispheres* (spheres with singularities of cone type $2\pi/n$): given an orbisphere map f , its biset $B(f)$ is a complete invariant of isotopy; and conversely given an orbisphere biset B there is an orbisphere map f , unique up to isotopy, realizing B , see Theorem 7.9.

We end this article, in §8, with a description of $M(f, A, A)$ for some Thurston maps $f: (S^2, A) \hookrightarrow$, recasting the calculations of [1] and its interpretation via Teichmüller theory in the language of mapping class bisets.

The language of trees of groups and trees of bisets is particularly well suited to describe the dynamical operations of *tuning* and *renormalizing*. Consider a Thurston map $f: (S^2, A) \hookrightarrow$. Tuning refers to removing a neighbourhood of a periodic critical cycle from S^2 , and replacing the map locally by a polynomial of same degree. Cutting S^2 along the boundaries of the neighbourhoods yields a tree of groups decomposition of the fundamental group, and a tree of bisets decomposition of $B(f)$, and tuning amounts to replacing cyclic bisets (that of a map z^d for some $d \in \mathbb{N}$) by bisets of polynomials in the tree of bisets.

Renormalizing refers to considering a periodic component of a decomposition of S^2 and taking its first return map. In terms of graphs of groups, this amounts to taking an invariant subtree of a tensor power of the biset.

Branched coverings $f: (S^2, C) \rightarrow (S^2, A)$, and the biset (1) may be considered for $C = \emptyset$, and correspond to well-studied objects. When $C = \emptyset$, the biset $B(f)$ is just a right $\pi_1(S^2 \setminus A, *)$ -set of cardinality $\deg(f)$, namely an $\#A$ -tuple of permutations in $\deg(f)!$ satisfying some conditions and considered up to conjugation (see §2.1).

Two branched coverings $f, f': S^2 \rightarrow (S^2, A)$ are isomorphic as coverings (they are also called “Hurwitz equivalent”) if and only if $f \approx m \circ f'$ for a mapping class $m \in \mathbf{Mod}(S^2, A)$, if and only if $f' \in M(f, \emptyset, A)$. The problem of classifying Hurwitz equivalence classes for which the tuple of permutations has given cycle structure (its “Hurwitz passport”) has been extensively investigated, even when $\#A = 3$, see e.g. [23, Chapter 5]. From our perspective, this problem can be interpreted as a classification of mapping class bisets with given portrait, as subbisets of $K(S^2, A)$.

1.3. Notation. We continue with the notation of the previous articles in the series. In particular, we write \curvearrowright for group actions, and $S\downarrow$ for the symmetric group on S .

Concatenation of paths is written $\gamma\#\delta$ for “first γ , then δ ”; inverses of paths are written γ^{-1} .

If $\beta: [0, 1] \rightarrow X$ is a path and $f: Y \rightarrow X$ is a covering map, we denote by $\beta\uparrow_f^y$ the f -lift of the path β starting at $y \in f^{-1}(\beta(0))$.

We write \approx for isotopy or homotopy of paths, maps etc, \sim for conjugacy or combinatorial equivalence, and \cong for isomorphism of algebraic objects.

The main objects of study are *marked*, or *punctured* spheres. There is no fundamental difference between $S^2 \setminus A$ and a pair (S^2, A) with $A \subset S^2$; it is usually more convenient to keep the points in A while remembering that they have a special status (for example, they are frozen by homotopies), but when marked spheres are cut along multicurves the boundary components appear topologically as punctures, not marked points. We therefore sometimes have to switch between these notations. We allow $A = \emptyset$, but forbid $\#A = 1$.

In all sections except §7, by $f: (S^2, C) \rightarrow (S^2, A)$ we denote an orientation preserving map between marked spheres, called a *sphere map*; see §2.1 for the precise definition. In §7 the map $f: (S^2, C) \rightarrow (S^2, A)$ will be allowed to reverse orientation.

For a small sphere $S \subset (S^2, A)$, cut by curves $\mathcal{C}_1, \dots, \mathcal{C}_m \subset S^2 \setminus A$, we denote by \overline{S} its topological closure, namely $S \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$, and by \hat{S} the quotient of \overline{S} obtained by shrinking every boundary curve of \overline{S} to a point. It is a sphere marked by the image of $A \cap S$ and the boundary curves. Depending on context, we call either of S , \overline{S} and \hat{S} a *small sphere*.

A map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ with marked curves \mathcal{C}, \mathcal{D} means $f(C) \subseteq A$ and $f^{-1}(\mathcal{C}) = \mathcal{D}$, unless it is explicitly stated that $f^{-1}(\mathcal{C}) \supseteq \mathcal{D}$. A *small sphere map* of f is the restriction of f to a small sphere of (S^2, C, \mathcal{D}) . For a Thurston map $f: (S^2, A, \mathcal{C}) \hookrightarrow$, we always require $f^{-1}(\mathcal{C}) = \mathcal{C}$; we call *small Thurston maps* of f the first return maps f^e on periodic small spheres of (S^2, A, \mathcal{C}) , see Lemma-Definition 4.9. We reserve the terminology ‘ \mathcal{C} is f -invariant’ to mean $f^{-1}(\mathcal{C}) = \mathcal{C}$; this is sometimes called *completely invariant* in the literature.

We denote by $\mathbb{1}$ the identity map. By a slight abuse of notation, if $A \subset C$ then we denote by $\mathbb{1}: (S^2, C) \rightarrow (S^2, A)$ the identity map on S^2 which erases the marked points in $C \setminus A$.

1.4. Acknowledgments. We are grateful to Jean-Pierre Spaenlehauer for having helped in computing the algebraic correspondence in §8.2.2.

2. SPHERES

The previous article [3] in the series set up the general theory of bisets associated with maps between topological spaces. Much sharper results may be obtained in a more specific context, that of maps between punctured spheres.

Let us consider a topological sphere S^2 with a finite, ordered collection of marked points $A = \{a_1, \dots, a_n\} \subset S^2$. We allow $n = 0$, but expressly forbid $n = 1$. Choose a basepoint $* \in S^2 \setminus A$. The fundamental group $\pi_1(S^2 \setminus A, *)$ may then be presented in the following, explicit manner: choose for each $i \in \{1, \dots, n\}$ a loop γ_i in $S^2 \setminus A$ that starts at $*$, goes towards a_i , encircles it counterclockwise in a very small loop, and returns to $*$ along the same path. Make sure that all paths γ_i are disjoint except at their endpoints, and that they are arranged counterclockwise around $*$. We call the γ_i *Hurwitz generators*, or colloquially *lollipops*. We then have

$$(2) \quad G = \pi_1(S^2, A, *) := \pi_1(S^2 \setminus A, *) = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n \rangle.$$

Indeed, if one cuts S^2 open along paths from the basepoint $*$ to the marked points a_i along the beginnings of the γ_i , one is left with a disk, whose perimeter gives the unique relation of G .

A cycle (loop without basepoint) in $S^2 \setminus A$ is represented by a conjugacy class in $G := \pi_1(S^2 \setminus A, *)$. The group G comes with extra data: the collection $\{\gamma_1^G, \dots, \gamma_n^G\}$ of conjugacy classes, called *peripheral conjugacy classes*, defined by the property that γ_i^G represents can be homotoped to a loop circling a_i once counterclockwise.

We use the notation $\beta \approx_A \gamma$ to mean that curves β, γ are isotopic relative to their endpoints and to A ; in other words, that $\beta \# \gamma^{-1}$ is trivial in the fundamental group $\pi_1(S^2, A, \beta(0))$.

Definition 2.1 (Sphere groups). A *sphere group* is a tuple $(G, \Gamma_1, \dots, \Gamma_n)$ consisting of a group and $n \neq 1$ conjugacy classes Γ_i in G , such that G admits a presentation as in (2) for some choice of $\gamma_i \in \Gamma_i$. The classes $\Gamma_1, \dots, \Gamma_n$ are called the *peripheral conjugacy classes* of G . \triangle

In other words, G is a free group of rank $n - 1$, with n distinguished conjugacy classes Γ_i satisfying a certain relation. If $n = 0$, then G is a trivial group. For every $d \in \mathbb{N}$ we denote by Γ_i^d the subset $\{g^d \mid g \in \Gamma_i\}$. We also set $\Gamma_i^+ = \bigcup_{d \geq 1} \Gamma_i^d$. By the remarks above,

Lemma 2.2. *Let (S^2, A) be a marked sphere, and choose $* \in S^2 \setminus A$. Then $\pi_1(S^2 \setminus A, *)$ is a sphere group. Conversely, given a sphere group (G, Γ_i) there exists a sphere, unique up to homeomorphism, whose fundamental group is isomorphic to G and whose peripheral conjugacy classes are $\{\Gamma_i\}$. \square*

From the perspective of a computer, a sphere group can be represented in two different manners. It may be considered as just a number ‘ n ’, and its elements are words in the symbols $\pm 1, \dots, \pm n$, subject to appropriate reduction rules. It may also be considered as a collection of n reduced words $\gamma_1, \dots, \gamma_n$ in the standard free group F_{n-1} on $n - 1$ generators, and its elements are in bijection with reduced words in F_{n-1} ’s generators. These two forms are obviously equivalent, and both have their advantages in terms of implementation.

2.1. Maps between spheres. Let $\mathbf{Mod}(S^2, A)$ denote the pure mapping class group of (S^2, A) , namely the set of isotopy classes of homeomorphisms $S^2 \hookrightarrow S^2$ fixing A pointwise. Given a choice of basepoint $* \in S^2 \setminus A$, consider the fundamental

group $G = \pi_1(S^2 \setminus A, *)$. Then every homeomorphism $\phi: (S^2, A) \hookrightarrow$ induces an isomorphism $G \rightarrow \pi_1(S^2 \setminus A, \phi(*))$; choosing a path ℓ from $*$ to $\phi(*)$ in $S^2 \setminus A$ leads to an automorphism of G defined by $[\gamma] \mapsto [\ell \# (\phi \circ \gamma) \# \ell^{-1}]$, which is well-defined up to inner automorphisms, and yields a homomorphism $\mathbf{Mod}(S^2, A) \rightarrow \text{Out}(G)$. The classical Dehn-Nielsen-Baer Theorem asserts that this map is injective, and describes its image:

Theorem 2.3 (Dehn-Nielsen-Baer, see [15, Theorem 8.8]). *Let $(G, \Gamma_1, \dots, \Gamma_n)$ be a sphere group. Then the natural map $\mathbf{Mod}(S^2, A) \rightarrow \text{Out}(G)$ is injective, and its image in $\text{Out}(G)$ consists of those automorphisms of G that map each Γ_i to itself.* \square

Definition 2.4. The *mapping class group* $\mathbf{Mod}(G)$ of a sphere group G as in Definition 2.1 is the group of outer automorphisms of G that preserve all peripheral conjugacy classes of G . \triangle

For calculations, it is useful to know that $\mathbf{Mod}(G)$ is generated by *Dehn twists*: for example, the automorphisms $\tau_{i,j}$, for $1 \leq i < j < n$, that map γ_k to $\gamma_k^{\gamma_i \gamma_{i+1} \dots \gamma_j}$ for $k \in \{i, \dots, j\}$ while fixing the other generators, see [25]. Geometrically, $\tau_{i,j}$ is a homeomorphism that acts as a “Chinese burn” on an annulus surrounding the points a_i, \dots, a_j and acts trivially elsewhere.

A *sphere map* $f: (S^2, C) \rightarrow (S^2, A)$ between marked spheres is a branched covering between the underlying spheres, locally modelled at $c \in S^2$ in oriented complex charts by $z \mapsto z^{\deg_c(f)}$ for some integer $\deg_c(f) \geq 1$, and such that A contains $f(C)$ and all $f(c)$ for which $\deg_c(f) > 1$. Those $c \in S^2$ with $\deg_c(f) > 1$ are called *critical points*, and their images $f(c)$ are called *critical values*. The map f is a *covering* if $C = f^{-1}(A)$. The restriction $f|_C: C \rightarrow A$ is called the *portrait* of $f: (S^2, C) \rightarrow (S^2, A)$. Two sphere maps $f_0, f_1: (S^2, C) \rightarrow (S^2, A)$ are *isotopic* if there is a continuous path $(f_t: (S^2, C) \rightarrow (S^2, A))_{t \in [0,1]}$ of sphere maps connecting them.

2.1.1. *Sphere bisets.* This notion of maps admits a precise algebraic translation, to which we turn now. Consider a sphere map $f: (S^2, C) \rightarrow (S^2, A)$; we may interpret it as a correspondence

$$(3) \quad (S^2, C) \xleftarrow{1} (S^2, f^{-1}(A)) \xrightarrow{f} (S^2, A)$$

such that $(S^2, C) \xleftarrow{1} (S^2, f^{-1}(A))$ forgets points in $f^{-1}(A) \setminus C$. Note that $(S^2, C) \xleftarrow{1} (S^2, f^{-1}(A))$ is not a sphere map (unless $C = f^{-1}(A)$), however its inverse $(S^2, C) \xrightarrow{1} (S^2, f^{-1}(A))$ is a sphere map. In [3, §4.1] we gave a general definition for the biset of a correspondence; applied to (3) this definition takes the following form (see [3, Equation (15)])

$$(4) \quad B(f, \dagger, *) = \{\delta: [0, 1] \rightarrow S^2 \setminus C \mid \delta(0) = \dagger, f(\delta(1)) = *\} / \approx_C,$$

and with left and right actions of $\pi_1(S^2, C, \dagger)$ and $\pi_1(S^2, A, *)$ given by preconcatenation, respectively postconcatenation through lifting by f . If $(S^2, C) = (S^2, A)$, then it will be usually assumed that $\dagger = *$; thus (4) is a $\pi_1(S^2, A, *)$ -biset. The biset (4) possesses extra properties coming from the peripheral conjugacy classes of the sphere groups $\pi_1(S^2, C, \dagger)$ and $\pi_1(S^2, A, *)$; they will be described in Definition 2.6 and Lemma 2.7.

Hurwitz describes in [19] an elegant classification of branched self-coverings $S^2 \hookrightarrow$ in terms of *admissible n -tuples* of permutations $(\sigma_1, \dots, \sigma_n)$ in $d!$. Such an n -tuple

is admissible if $\sigma_1 \cdots \sigma_n = 1$, the group $\langle \sigma_1, \dots, \sigma_n \rangle$ acts transitively on $\{1, \dots, d\}$, and the cycle lengths of the permutations $\gamma_1, \dots, \gamma_n$ satisfy

$$(5) \quad \sum_{i=1}^n \sum_{\substack{c \text{ cycle} \\ \text{of } \sigma_i}} (\text{length}(c) - 1) = 2d - 2.$$

Hurwitz gives a bijection between degree- d branched self-coverings $S^2 \hookrightarrow$ with critical values contained in $\{a_1, \dots, a_n\}$, viewed up to isomorphism of coverings, and admissible n -tuples in $d\downarrow$, viewed up to global conjugacy by $d\downarrow$ and the “mapping class group action”. For the latter, set $G = \pi_1(S^2 \setminus A, *)$ and identify $(\sigma_1, \dots, \sigma_n)$ with the homomorphism $\pi: G \rightarrow d\downarrow$ given by $\gamma_i \mapsto \sigma_i$, and let the mapping class group $\mathbf{Mod}(G)$ act on π by precomposition. Condition (5) amounts to the statement that the Euler characteristic of the cover is $2 = \chi(S^2)$.

Definition 2.5 (Multiset of lifts [3, §2.6]). Let ${}_H B_G$ be a left-free biset, and contract every left orbit of B to a point by considering $\{\cdot\} \otimes_H B$. Consider $g \in G$. Then $\{\cdot\} \otimes_H B$ decomposes into orbits $S_1 \sqcup \dots \sqcup S_\ell$ under the right action of g , of respective cardinalities d_1, \dots, d_ℓ ; and for all $i = 1, \dots, \ell$, choosing $s_i \in B$ with $\{\cdot\} \otimes_H s_i \in S_i$ there are elements $h_i \in H$ with $h_i s_i = s_i g^{d_i}$. The multiset $\{(d_i, h_i^H) \mid i = 1, \dots, \ell\}$ consisting of degrees and conjugacy classes in H is independent of the choice of the s_i , and depends only on the conjugacy class of g ; it is called the *multiset of lifts* of g^G . \triangle

If B be the biset of a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ and g^G be interpreted as a simple closed curve, then the multiset $\{(h_i^H) \mid i = 1, \dots, \ell\}$ is, up to homotopy, the set $f^{-1}(g^G)$, and the d_i are the degrees with which f maps the corresponding closed curves onto g^G .

Definition 2.6 (Sphere bisets). Consider (H, Δ_j) and (G, Γ_i) two sphere groups. A *sphere biset* is an H - G -biset B such that the following hold:

- (SB₁) B is left-free and right-transitive;
- (SB₂) choose representatives $\gamma_1 \in \Gamma, \dots, \gamma_n \in \Gamma_n$; then the permutations of $\{\cdot\} \otimes_H B$ induced by the right action of $\gamma_1, \dots, \gamma_n$ satisfy (5);
- (SB₃) the multiset of all lifts of $\Gamma_1, \dots, \Gamma_n$ contains exactly once every Δ_j , all the other conjugacy classes being trivial.

By the last condition, to every peripheral conjugacy class Δ_j in H is associated a well-defined *degree* $\deg_{\Delta_j}(B) \in \mathbb{N}$ and conjugacy class $\Gamma_i =: B_*(\Delta_j)$, such that $(\deg_{\Delta_j}(B), \Delta_j)$ belongs to the lift of Γ_i . We define in this manner a map B_* from the peripheral conjugacy classes in H to those of G , called the *portrait* of B . \triangle

In case the peripheral conjugacy classes of G, H are indexed as $(\Gamma_a)_{a \in A}$ and $(\Delta_c)_{c \in C}$ respectively, we write $B_*(c) = a$ rather than $B_*(\Delta_c) = \Gamma_a$, defining in this manner a map $B_*: C \rightarrow A$. It is easy to see that if B be the biset of $f: (S^2, C) \rightarrow (S^2, A)$ and Γ_a and Δ_c denote the peripheral conjugacy classes around $a \in A$ and $c \in C$ respectively, then $\deg_{\Delta_c}(B) = \deg_c(f)$, and $f(c) = a$ if and only if $B_*(c) = a$.

Lemma 2.7. *The biset defined in (4) of a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ is a sphere $\pi_1(S^2, C, \dagger) \text{-} \pi_1(S^2, A, *)$ -biset.*

Proof. The biset $B(f)$ is clearly left-free. Consider $b_1, b_2 \in B(f)$. Up to homotopy we may assume that b_1, b_2 , and \dagger are away from $f^{-1}(A)$. Set $g := f(b_1^{-1}) \# f(b_2) \in \pi_1(S^2, A, *)$. Then $b_1 g = b_2$; this verifies Condition (SB₁) of Definition 2.6.

Consider a peripheral conjugacy class Γ , say around a puncture $a \in A$. Let $\{(d_i, h_i^H) \mid i = 1, \dots, \ell\}$ be the multiset of lifts of Γ . We may enumerate $f^{-1}(a)$ as $\{c_1, c_2, \dots, c_\ell\}$ such that f maps c_i to a with degree d_i and such that h_i^H is the peripheral conjugacy class around c_i , respectively the trivial class, if $c_i \in C$, respectively $c_i \notin C$. This verifies Condition (SB₃) of Definition 2.6. Since $f^{-1}(A)$ contains $2d - 2$ critical points counting with multiplicities, we have (5); this is Condition (SB₂). \square

2.1.2. Equivalence between sphere maps and bisets. Kameyama essentially proves in [20, Theorem 3.6] that to an isomorphism class of sphere bisets corresponds a unique isotopy class of sphere maps. In fact, we give a bijection between isotopy classes of maps and isomorphism classes bisets, extending at the same time the Dehn-Nielsen-Baer Theorem 2.3 to non-invertible maps:

Theorem 2.8. *Let $f_0, f_1: (S^2, C) \rightarrow (S^2, A)$ be sphere maps, and consider $H := \pi_1(S^2 \setminus C, \dagger)$ and $G := \pi_1(S^2 \setminus A, *)$ for choices of $\dagger \in S^2 \setminus C$ and $*$ in $S^2 \setminus A$. Then $B(f_0)$ is a sphere H - G -biset, and $B(f_0) \cong B(f_1)$ if and only if $f_0 \approx f_1$.*

Conversely, for every sphere H - G -biset B there exists a sphere map $f: (S^2, C) \rightarrow (S^2, A)$, unique up to isotopy, such that $B \cong B(f)$.

The proof of Theorem 2.8 appears below, after some preparation. Consider a sphere H - G -biset B . Choose $b \in B$, and let G_b be the stabilizer of $\{\cdot\} \otimes b$ in the G -set $\{\cdot\} \otimes_H B$. It is a subgroup of G of index equal to the degree of B . Then B naturally splits as

$$(6) \quad {}_H B_G \cong HbG_b \otimes_{G_b} G_G.$$

This splitting is the algebraic counterpart of (3), and we use it to give, along the way, the structure of a sphere group to G_b :

Lemma 2.9. *Suppose that ${}_H B_G$ is the biset of a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ as in (4) and with $G = \pi_1(S^2 \setminus A, *)$ and $H = \pi_1(S^2 \setminus C, \dagger)$. Consider $b \in B$ and let $*'$ be the endpoint of b .*

*Then $\pi_1(S^2, f^{-1}(A), *')$ is identified via $f_*: \pi_1(S^2, f^{-1}(A), *') \rightarrow \pi_1(S^2, A, *)$ with G_b , and via this identification the $\pi_1(S^2, f^{-1}(A), *')$ - G -biset of $f: (S^2, f^{-1}(A)) \rightarrow (S^2, A)$ is isomorphic to ${}_{G_b} G_G$ while the H - $\pi_1(S^2, f^{-1}(A), *')$ -biset of $(S^2, C) \xrightarrow{1} (S^2, f^{-1}(A))$ is isomorphic to HbG_b .*

*Moreover, via the identification of G_b with $\pi_1(S^2, f^{-1}(A), *')$ the peripheral conjugacy classes of G_b are $(\Xi_{i,j})_{i,j}$ constructed as follows. Let $\Gamma_1, \dots, \Gamma_n$ be the peripheral conjugacy classes of G . Then for each Γ_i there is a unique decomposition*

$$(7) \quad \Gamma_i^+ \cap G_b = \Xi_{i,1}^+ \sqcup \Xi_{i,2}^+ \sqcup \dots \sqcup \Xi_{i,s}^+$$

such that every $\Xi_{i,j}$ is a conjugacy class of G_b . Assuming $\Xi_{i,j}$ is generated by $\gamma_{i,j}^{d(i,j)}$ with $\gamma_{i,j} \in \Gamma_i$, we let $\{(d(i,j), \Xi_{i,j})\}$ be the multiset of lifts of Γ_i via ${}_{G_b} G_G$.

Proof. Since $f: S^2 \setminus f^{-1}(A) \rightarrow S^2 \setminus A$ is a covering map, $\pi_1(S^2, f^{-1}(A), *')$ is identified with G_b via f_* and the biset of $f: (S^2, f^{-1}(A)) \rightarrow (S^2, A)$ is ${}_{G_b} G_G$. It follows immediately from (4) that the H - $\pi_1(S^2, f^{-1}(A), *')$ -biset of $(S^2, C) \xrightarrow{1} (S^2, f^{-1}(A))$ is HbG_b .

Let us prove the claims concerning $(\Xi_{i,j})_{i,j}$. Suppose that Γ_i is a peripheral conjugacy class around $a_i \in A$, suppose that $f^{-1}(a_i) = \{c_{i,1}, c_{i,2}, \dots, c_{i,s}\}$ and that f has degree $d(i,j)$ at $c_{i,j}$, and suppose that $\Xi_{i,j}$ is the peripheral conjugacy class of

$\pi_1(S^2, f^{-1}(A), *)$ around $c_{i,j}$. Observe that $\Xi_{i,j}$ are pairwise disjoint as peripheral conjugacy classes around different points in $f^{-1}(A)$. Therefore, $\Xi_{i,j}^+$ are pairwise disjoint. Then f_* identifies $\Xi_{i,1}^+ \sqcup \Xi_{i,2}^+ \sqcup \dots \Xi_{i,s}^+$ with $\Gamma_i^+ \cap G_b$ and we get (7). If $d(i,j)$ be the local degree of f at $c_{i,j}$, then $f_*(\Xi_{i,j})$ is generated by $\gamma_{i,j}^{d(i,j)}$ with $\gamma_{i,j} \in \Gamma_i$. \square

Lemma 2.10. *Consider a sphere H - G -biset B and choose $b \in B$. Endow G_b with the sphere structure given in Lemma 2.9: assuming that $\Gamma_1, \dots, \Gamma_n$ are peripheral conjugacy classes of G , the peripheral conjugacy classes of G_b are $(\Xi_{i,j})_{i,j}$ specified by (7). Then in the decomposition (6) the bisets HbG_b and ${}_{G_b}G_G$ are sphere bisets.*

Proof. Observe that $\{\cdot\} \otimes_H B \cong \{\cdot\} \otimes_{G_b} G$ as right G -sets; thus the action of G on $\{\cdot\} \otimes_{G_b} G$ satisfies the Hurwitz condition (5). Write $G = \pi_1(S^2 \setminus A, *)$ as a sphere group; then by Hurwitz's theorem [19], the cover $(S^2 \setminus A)/G_b$ of $S^2 \setminus A$ associated with $G_b \leq G$ is a finitely-punctured sphere. By Lemmas 2.7 and 2.9 the group G_b is a sphere group with peripheral conjugacy classes $(\Xi_{i,j})_{i,j}$ specified by (7) and, moreover, ${}_{G_b}G_G$ is a sphere biset.

Observe that the multiset of the lifts of Γ_i via ${}_HB_G$ is the multiset of the lifts via HbG_b of the lifts via ${}_{G_b}G_G$ of Γ_i with the corresponding degrees being multiplied. Since HbG_b has degree one, the multiset of the lifts of $\Xi_{i,j}$ is a singleton $\{(1, \tilde{\Xi}_{i,j})\}$. Therefore, the multiset of the lifts of Γ_i is $\{(d(i,j), \tilde{\Xi}_{i,j})\}$ with $d(i,j)$ as in Lemma 2.9. This verifies Conditions (SB₂) and (SB₃) of Definition 2.6; Condition (SB₁) is immediate. \square

Proof of Theorem 2.8. If $f_0 \approx f_1$, say via a path $(f_t: (S^2, C) \rightarrow (S^2, A))_{t \in [0,1]}$, then $B(f_0) \cong B(f_1)$ because $B(f_t)$, being a discrete object, remains constant along the isotopy.

Assume $B(f_0) \cong B(f_1)$; say $\beta: B(f_0) \rightarrow B(f_1)$ is a biset isomorphism. Let us construct a homeomorphism $\psi: (S^2, C) \hookrightarrow$ such that $f_0 = \psi f_1$ and such that $B(\psi) \cong {}_HH_H$. Then using Theorem 2.3 we will get $\psi \approx 1$, and therefrom $f_0 \approx f_1$.

Choose $b \in B(f_0)$ and decompose ${}_HB(f_0)_G \cong HbG_b \otimes_{G_b} G_G$ and ${}_HB(f_1)_G \cong H\beta(b)G_{\beta(b)} \otimes_{G_{\beta(b)}} G_G$ as in (6). Since β is an isomorphism we have $G_b = G_{\beta(b)}$, so ${}_{G_b}G_G = {}_{G_{\beta(b)}}G_G$. By Lemma 2.9 the bisets of $f_0: (S^2, f_0^{-1}(A)) \rightarrow (S^2, A)$ and $f_1: (S^2, f_1^{-1}(A)) \rightarrow (S^2, A)$ are respectively isomorphic to ${}_{G_b}G_G$ and ${}_{G_{\beta(b)}}G_G$ after identifying $\pi_1(S^2, f_0^{-1}(A))$ and $\pi_1(S^2, f_1^{-1}(A))$ with G_b and $G_{\beta(b)}$ as in Lemma 2.9. Therefore, $f_0: (S^2, f_0^{-1}(A)) \rightarrow (S^2, A)$ and $f_1: (S^2, f_1^{-1}(A)) \rightarrow (S^2, A)$ are covering maps associated with the same subgroup of G ; thus the map $\psi := f_0 \circ f_1^{-1}: (S^2, f_0^{-1}(A)) \rightarrow (S^2, f_1^{-1}(A))$ is a well defined homeomorphism specified so that ψ (the endpoint of b) = the endpoint of $\beta(b)$. Finally

$${}_HB(\psi: (S^2, C) \hookrightarrow)_H = HbG_b \otimes_{G_b = G_{\beta(b)}} G_{\beta(b)} \beta(b)H \cong {}_HH_H.$$

Let us now prove the second part of the theorem. Decompose ${}_HB_G \cong HbG_b \otimes_{G_b} G_G$ as in (6). By Hurwitz's theorem [19] the cover of $S^2 \setminus A$ associated with $G_b \leq G$ is a finitely punctured sphere; write this cover as $f: S^2 \setminus f^{-1}(A) \rightarrow S^2 \setminus A$. As in Lemma 2.9 we identify G_b with $\pi_1(S^2, f^{-1}(A))$ and we denote by $(\Xi_a)_{a \in f^{-1}(A)}$ the peripheral conjugacy classes of $G_b \cong \pi_1(S^2, f^{-1}(A))$.

By Lemma 2.10 the biset HbG_b is a sphere biset. Since HbG_b has a single left orbit, for every $a \in f^{-1}(A)$ the multiset of the lifts of Ξ_a is a singleton $\{(1, \tilde{\Xi}_a)\}$.

Then the homomorphism $\iota: G_b \rightarrow H$ given by $bg = \iota(g)b$ has the property that $\iota(\Xi_a) = \tilde{\Xi}_a$; i.e. ι “forgets” all Ξ_a with trivial $\tilde{\Xi}_a$.

Let $\mathbb{1}: (S^2, f^{-1}(A)) \rightarrow (S^2, C')$ be the map forgetting all $a \in f^{-1}(A)$ with trivial $\tilde{\Xi}_a$. Since the biset of this map is naturally identified with HbG_b , we obtain a natural isomorphism of $\pi_1(S^2, C')$ with H , and therefore of C' with C . Using these identifications, the biset of $(S^2, C') \xrightarrow{\mathbb{1}} (S^2, f^{-1}(A)) \xrightarrow{f} (S^2, A)$ is isomorphic to ${}_HB_G$.

It follows from Theorem 2.3 that there is a map, unique up to isotopy, identifying (S^2, C) with (S^2, C') and such that the biset of this map is isomorphic to ${}_HH_H$. This finishes the construction of a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ with the biset ${}_HB_G$. \square

2.2. Thurston maps. Consider a branched self-covering $f: (S^2, A) \curvearrowright$. This means $f(A) \subseteq A$ and A contains the critical values of f . In particular, A contains the *post-critical set* of f ,

$$P_f = \bigcup_{k \geq 1} f^k(\text{critical points of } f).$$

On the other hand, A could also contain fixed points of f , or more generally preperiodic points together with their forward orbit. If A is finite, then f is called a *Thurston map*.

Let $f: (S^2, A) \curvearrowright$ be a Thurston map. In particular, f induces a map $A \curvearrowright$. Then its portrait $f: A \curvearrowright$ is a finite dynamical system, sometimes called its *critical portrait*.

Definition 2.11 (Combinatorial equivalence of maps). Let $f_0: (S^2, A_0) \curvearrowright$ and $f_1: (S^2, A_1) \curvearrowright$ be two Thurston maps. We say that f_0 and f_1 are *combinatorially equivalent*, aka “Thurston-equivalent”, if there exists a path of Thurston maps $(f_t: (S^2, A_t) \curvearrowright)_{t \in [0,1]}$ connecting f_0 to f_1 . In that case, we write $f \sim g$. \triangle

There is an equivalent formulation in terms of isotopy:

Lemma 2.12. *Two Thurston maps $f_0: (S^2, A_0) \curvearrowright$ and $f_1: (S^2, A_1) \curvearrowright$ are combinatorially equivalent if and only if there exists homeomorphisms $\phi_0, \phi_1: (S^2, A_0) \rightarrow (S^2, A_1)$ with $\phi_0 \circ f_0 = f_1 \circ \phi_1$ and ϕ_0 isotopic to ϕ_1 rel A_0 :*

$$\begin{array}{ccc} (S^2, A_0) & \xrightarrow{\phi_1} & (S^2, A_1) \\ f_0 \downarrow & \curvearrowright & \downarrow f_1 \\ (S^2, A_0) & \xrightarrow{\phi_0} & (S^2, A_1) \end{array}$$

commutes up to isotopy rel A_0 .

Proof. Given a path (f_t) of Thurston maps connecting f_0 to f_1 , factor each map f_t as $f_t = \lambda_t \circ f_0 \circ \rho_t^{-1}$ for homeomorphisms $\lambda_t, \rho_t: (S^2, A_0) \rightarrow (S^2, A_t)$ depending continuously on t , with $\lambda_0 = \rho_0 = \mathbb{1}$. Define then $\phi_t = \rho_1 \circ \rho_{1-t}^{-1} \circ \lambda_{1-t}$, and note that $\phi_t: (S^2, A_0) \rightarrow (S^2, A_1)$ is a homeomorphism with $\phi_0 = \lambda_1$ and $\phi_1 = \rho_1$ so $\phi_0 \circ f_0 = f_1 \circ \phi_1$. Furthermore, $(\phi_t)_{t \in [0,1]}$ is an isotopy rel A_0 from ϕ_0 to ϕ_1 .

Conversely, let $\phi_0, \phi_1: (S^2, A_0) \rightarrow (S^2, A_1)$ be isotopic homeomorphisms, and let $(\phi_t)_{t \in [0,1]}$ be an isotopy rel A_0 between them. Let $(\lambda_t)_{t \in [0,1]}$ be an isotopy (possibly non-constant on A_0) from $\lambda_0 = \mathbb{1}$ to $\lambda_1 = \phi_0$. Define $A_t := \lambda_t(A_0)$ and

$f_t := \lambda_t \circ f_0 \circ \phi_t^{-1} \circ \phi_0 \circ \lambda_t^{-1}$, and note $f_t(A_t) \subseteq A_t$, so f_t is a Thurston map along an isotopy from f_0 to f_1 . \square

Corollary 2.13 (Kameyama [20, Corollary 3.7]; see also [27, Theorem 6.5.2]). *Two Thurston maps $f: (S^2, A) \hookrightarrow (S^2, C)$ and $g: (S^2, C) \hookrightarrow (S^2, A)$ with $|C|, |A| \geq 2$ are combinatorially equivalent if and only if their bisets $B(f)$ and $B(g)$ are conjugate.*

Proof. If $f: (S^2, A) \hookrightarrow (S^2, C)$ and $g: (S^2, C) \hookrightarrow (S^2, A)$ are combinatorially equivalent, then by Lemma 2.12 there exists a homeomorphism $\phi: (S^2, A) \rightarrow (S^2, C)$ such that f and $\phi^{-1} \circ g \circ \phi$ are isotopic rel A ; so $B(f)$ and $B(\phi) \otimes B(g) \otimes B(\phi)^\vee$ are isomorphic by Theorem 2.8, and $B(f)$ and $B(g)$ are conjugate.

Conversely, if ${}_G B(f)_G$ and ${}_H B(g)_H$ are conjugate with respective sphere groups $G = \pi_1(S^2 \setminus A, *)$ and $H = \pi_1(S^2 \setminus C, \dagger)$, let $\varphi: G \rightarrow H$ be a sphere group isomorphism such that the bisets $B(f)$ and $B_\varphi \otimes B(g) \otimes B_\varphi^\vee$ are isomorphic. By Theorem 2.3, the homomorphism φ may be (uniquely) realized as ϕ_* for a homeomorphism $\phi: (S^2, A) \rightarrow (S^2, C)$, so f and $\phi^{-1} \circ g \circ \phi$ are isotopic rel A by Theorem 2.8, so f and g are combinatorially equivalent by Lemma 2.12. \square

3. DECOMPOSITIONS OF SPHERE BISETS

We now explore how spheres and sphere bisets can be decomposed into simpler components. As before, we describe side by side the topology and its associated group theory.

We will use the following conventional notations. Two closed curves ℓ, ℓ' in $S^2 \setminus A$ are *homotopic rel A* if their parameterizations $\ell, \ell': S^1 \rightarrow S^2 \setminus A$ are homotopic rel A . Given a subset $T \subset S^2$, we say that T is *homotopic rel A* to a point $x \in S^2$ (or T is *contractible* to x) if there is a homotopy $h: T \times [0, 1] \rightarrow S^2$ rel A , namely $h(-, 0) = 1$ and $h(y, t) = h(y, 0)$ for all $y \in T \cap A$, and $h(-, t) \rightarrow x$ as $t \rightarrow 1$. Note that if T is homotopic to x , then either $T \cap A \subseteq \{x\}$.

Finally, we say that $T \subset S^2$ is *homotopic to a curve ℓ* if there is a homotopy $h: T \times [0, 1] \rightarrow S^2$ rel A and a curve $\ell' \subset T$ such that $h(-, 0) = 1$, $h(T, 1) \subset \ell$, and the restriction of h to $\ell' \times [0, 1]$ is a homotopy rel A between ℓ' and ℓ .

3.1. Multicurves. Let (S^2, A) be a sphere. A *multicurve* \mathcal{C} on (S^2, A) is a collection of non-trivial, non-peripheral, mutually non-homotopic, non-intersecting, simple closed curves on $S^2 \setminus A$.

Psychologically, a sphere with multicurve (S^2, A, \mathcal{C}) should be thought of as a sphere on which the curves \mathcal{C} are extremely short, so that the sphere looks more like a cactus of the genus *opuntia*.

If (S^2, A, \mathcal{C}) be a noded sphere, one may cut S^2 along \mathcal{C} , and shrink the boundary components to punctures. Algebraically, this amounts to the following. Each curve in \mathcal{C} may be expressed as a conjugacy class Γ in $G := \pi_1(S^2, A)$. Choose in each $\Gamma \in \mathcal{C}$ a representative $t_\Gamma \in \Gamma$. Then G decomposes as a tree of groups, with one vertex per component S of $S^2 \setminus \mathcal{C}$ and associated vertex group $\pi_1(S)$, and one edge per curve $\Gamma \in \mathcal{C}$ with associated edge group $\langle t_\Gamma \rangle$. The underlying graph is a tree.

So as to follow [3, Definition 3.6], we consider rather the barycentric subdivision of the above tree of groups:

Definition 3.1 (Tree of groups of a multicurve). Let (S^2, A, \mathcal{C}) be a sphere given with a multicurve \mathcal{C} . Its associated *sphere tree of groups* \mathfrak{X} is defined as follows.

Consider the finite 1-dimensional cover [3, Definition 3.4] of (S^2, A) by components of \mathcal{C} and the set \mathcal{S} of closures of components of $S^2 \setminus \mathcal{C}$; call the former *curves* and the latter *small spheres*. The underlying graph of \mathfrak{X} has one vertex per curve and one per small sphere. It has four edges per curve, with an edge connecting each curve to its two neighbouring spheres and back. Thus $V = \mathcal{C} \sqcup \mathcal{S}$ is the vertex set, and $E = \{(v, w) \in \mathcal{C} \times \mathcal{S} \sqcup \mathcal{S} \times \mathcal{C} \mid v \cap w \neq \emptyset\}$ is the edge set. We have $(v, w)^- = v$ and $(v, w) = (w, v)$.

Consider a vertex $v \in V$; it is represented by a subset S_v of S^2 , and is either a curve or a small sphere. Choose a basepoint $*_v \in S_v \setminus A$, and for each edge (v, w) choose a path $\ell_{v,w}$ in $(S_v \cup S_w) \setminus A$ from $*_v$ to $*_w$, assuming $\ell_{v,w}^{-1} = \ell_{w,v}$. The group associated to the vertex v is the fundamental group $G_v = \pi_1(S_v \setminus A, *_v)$. The group associated with the edge (v, w) is G_v if $v \in \mathcal{C}$ and is G_w if $w \in \mathcal{C}$. The homomorphism $G_{(v,w)} \rightarrow G_{(w,v)}$ is the identity. The homomorphism $G_{(v,w)} \mapsto G_v$ is the identity if $v \in \mathcal{C}$ and is $\gamma \mapsto \ell_{v,w} \# \gamma \# \ell_{w,v}$ otherwise. \triangle

In particular, the groups associated with small spheres are sphere groups (there is no distinction, from their point of view, between boundary components in A or in \mathcal{C}), and the vertex groups associated with curves are infinite cyclic (they could be thought as sphere groups of $S^2 \setminus \{0, \infty\}$), as are the edge groups. The underlying graph is a tree. The van Kampen theorem, see [3, Theorem 3.7], asserts that the fundamental group of (S^2, A) is isomorphic to the fundamental group of \mathfrak{X} . We recall from [3, Lemma 4.9] that a tree of groups is defined uniquely up to congruence (see [3, Definition 3.29]).

For a small sphere $\bar{S} \subset (S^2, A)$, we write \hat{S} for the quotient of \bar{S} in which every boundary curve is shrunk to a point; so \hat{S} is a topological sphere, marked by the image of A and the boundary curves.

3.1.1. Algebraic multicurves. Recall from [3, §3.2] that the *barycentric subdivision* of a graph $\mathfrak{X} = V \sqcup E$ is a new graph $\mathfrak{X}' = V' \sqcup E'$ with vertex set $V' = \mathfrak{X}/\{x = \bar{x}\} = V \sqcup E/\{x = \bar{x}\}$ and edge set $E' = E \times \{+, -\}$; for $e \in E$ and $\varepsilon \in \{\pm 1\}$, set $(e, \varepsilon)^\varepsilon = e^\varepsilon$ and $(e, \varepsilon)^{-\varepsilon} = [e]$ and $(\overline{e, \varepsilon}) = (\bar{e}, -\varepsilon)$. Let us say that a vertex $v \in V'$ is *old* if $v \in V$ and is *new* if $v \in E/\{x = \bar{x}\}$. Old and new vertices form a bipartite structure of \mathfrak{X}' . We are now ready to state an algebraic counterpart of Definition 3.1; the equivalence of the objects is proven in Lemma 3.3.

Definition 3.2 (Sphere groups and algebraic multicurves). *A stable sphere tree of groups is*

- (1) a tree \mathfrak{X} that is the barycentric subdivision of a smaller tree; old vertices of \mathfrak{X} are called *sphere vertices* while new vertices of \mathfrak{X} are called *curve vertices*;
- (2) a sphere group at every vertex, so that curve vertex groups are cyclic (thought of as $\pi_1(\hat{\mathbb{C}} \setminus \{0, \infty\})$) while sphere vertex groups have at least 3 peripheral conjugacy classes;
- (3) a cyclic group at every edge, which embeds into vertex groups by mapping a generator to an element of a peripheral conjugacy class, in such a manner that different edge groups attach to different peripheral conjugacy classes.

A peripheral conjugacy class in a vertex group G_v is *vacant* if it doesn't intersect the image of any edge group. Clearly, vacant peripheral classes in a sphere tree of groups \mathfrak{X} are identified with peripheral conjugacy classes in $\pi_1(\mathfrak{X})$.

Let G be a sphere group. An algebraic *multicurve* is a collection \mathcal{C} of distinct, non-peripheral conjugacy classes in G , such that there exists a decomposition of G as a sphere tree of groups with edges in bijection with \mathcal{C} , such that the edge group associated with $\Gamma \in \mathcal{C}$ is cyclic and generated by a representative $t_\Gamma \in \Gamma$. \triangle

The following extends Lemma 2.2 to spheres with multicurves:

Lemma 3.3. *Let (S^2, A) be a marked sphere with multicurve \mathcal{C} , and choose $* \in S^2 \setminus A$. Then the tree of groups decomposition \mathfrak{X} of $G = \pi_1(S^2 \setminus A, *)$ along \mathcal{C} is a sphere tree of groups as in Definition 3.2. Via this decomposition, peripheral conjugacy classes in G form vacant peripheral conjugacy classes in \mathfrak{X} while conjugacy classes of G encoding \mathcal{C} form non-vacant peripheral conjugacy classes of \mathfrak{X} .*

Conversely, to a sphere tree of groups corresponds a marked sphere with multicurve, which is unique up to homeomorphism.

Proof. It is easy to check that the tree of groups decomposition of $\pi_1(S^2 \setminus A, *)$ along \mathcal{C} satisfies Definition 3.2. Conversely, given a sphere tree of groups as in Definition 3.2, construct a sphere with multicurve by first realizing every small sphere vertex group as the fundamental group of a sphere with marked points; remove a small disk around each marked point whose corresponding peripheral class is not vacant, and attach cylinders, with a marked curve along their core, between these spheres by gluing their boundary to the boundary of the removed disks. The uniqueness of the obtained sphere follows from Theorem 2.3. \square

Note that there exists a general notion, that of “JSJ decomposition” of groups [30], over cyclic subgroups. One usually considers it for freely indecomposable groups; while, on the other hand, we apply it here to study bisets over decompositions of free groups over cyclic subgroups.

Theorem 3.4. *There is an algorithm that, given (G, Γ_i) a sphere group and \mathcal{C} a collection of conjugacy classes in G , computes a sphere tree of groups decomposition as in Definition 3.2 (if \mathcal{C} is an algebraic multicurve), or returns **fail** (if it is not).*

Proof. There are algorithms (see [9, 10]) that compute geometric intersection numbers of curves; and, in particular, whether a conjugacy class represents a simple closed curve.

Given a multicurve \mathcal{C} , apply such an algorithm to each curve g^G of \mathcal{C} , in turn. If g^G is a simple closed curve, then $\{1, \dots, n\}$ may be partitioned as $\{i_1, \dots, i_s\} \sqcup \{j_1, \dots, j_t\}$ in such a manner that g may be written as a product of conjugates of Hurwitz generators of G as

$$g = \gamma_{i_1}^{u_1} \cdots \gamma_{i_s}^{u_s} = (\gamma_{j_1}^{v_1} \cdots \gamma_{j_t}^{v_t})^{-1}$$

for $u_1, \dots, u_s, v_1, \dots, v_t \in G$. The partition $\{i_1, \dots, i_s\} \sqcup \{j_1, \dots, j_t\}$ is obtained by computing the image of g in $G/[G, G] \cong \mathbb{Z}^n/(1, \dots, 1)$ and writing it as $e_{i_1} + \dots + e_{i_s} = -(e_{j_1} + \dots + e_{j_t})$ with e_1, \dots, e_n the image of the standard basis of \mathbb{Z}^n .

The claimed decomposition of g exists for topological reasons: picture the sphere with $*$ at the north pole, all a_i on the equator, and all lollipop generators along meridians except for their little loop around a_i . Consider g as a closed curve on S^2 . There is then an isotopy of S^2 that brings g into a thin neighbourhood of the equator, turning once around it. Under the isotopy, every lollipop generators gets deformed to a conjugate of itself. The curve g is isotopic to the product of the original lollipop generators γ_i that lie above it, namely such that the curve g passes

below a_i as it revolves around S^2 . Once we know that such a decomposition exists, we can find it by enumerating all choices of $u_1, \dots, u_s, v_1, \dots, v_t$.

This shows how to split G as an amalgamated free product over \mathbb{Z} : the edge group \mathbb{Z} is generated by g , one vertex group is the sphere group G' generated by $\gamma'_1 = \gamma_{i_1}^{u_1}, \dots, \gamma'_s = \gamma_{i_s}^{u_s}, g^{-1}$ and relation $\gamma'_1 \cdots \gamma'_s g^{-1} = 1$, and the other vertex group G'' is similarly generated by g and the $\gamma''_1 = \gamma_{j_1}^{v_1}, \dots, \gamma''_t = \gamma_{j_t}^{v_t}$ with relation $\gamma''_1 \cdots \gamma''_t g = 1$.

The remaining curves in $\mathcal{C} \setminus \{g^G\}$ should now be rewritten in terms of the generators of G' or of G'' . If this is impossible, then the curves in \mathcal{C} are not disjoint. Otherwise, proceed inductively with G' and G'' , which have smaller complexity. \square

3.2. Invariant multicurves. Consider a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ and a multicurve \mathcal{C} in (S^2, A) . Since f is a covering away from A , there is a *lifted* multicurve $f^{-1}(\mathcal{C})$ in (S^2, C) .

Algebraically, f is represented as an H - G -biset $B(f)$, and \mathcal{C} is represented by conjugacy classes in G up to $g^G \leftrightarrow (g^{-1})^G$, which we denote by $g^{\pm G}$. The multiset of lifts, Definition 2.5, can be defined for g^{\pm} since the inverse of a lift is the lift of the inverse. The multiset of lifts of a multicurve under $B(f)$ is the f -preimage of the multicurve.

We consider more generally the situation of a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$, meaning \mathcal{D} is isotopic to a subset of $f^{-1}(\mathcal{C})$. For our purpose, the most valuable information about lifts of multicurves is contained in the *Thurston matrix*, see [3, (2)]: it is the linear operator $T_{f, \mathcal{C}}: \mathbb{Q}\mathcal{C} \rightarrow \mathbb{Q}\mathcal{D}$ given by

$$(8) \quad T_{f, \mathcal{C}}(\gamma) = \sum_{\substack{\varepsilon \in f^{-1}(\gamma) \\ \varepsilon \approx \delta \in \mathcal{D}}} \frac{1}{\deg(f \downarrow_{\varepsilon}: \varepsilon \rightarrow \gamma)} \delta.$$

Here by \deg one means the usual positive degree of f ; i.e. for $d \in \mathbb{Z}$ the degree of $z^d: \{|z| = 1\} \hookrightarrow$ is $|d|$.

If ${}_H B_G$ be a sphere biset, \mathcal{C} be a multicurve in G and \mathcal{D} be a multicurve in H , both represented as collections of conjugacy classes, then by [3, §2.6] there is a linear operator $T_{B, \mathcal{C}}: \mathbb{Q}\mathcal{C} \rightarrow \mathbb{Q}\mathcal{D}$, which coincides with (8) in case $B = B(f)$.

In Proposition 6.9 we shall later interpret the Thurston matrix as giving the structure of abelian subbisets for the mapping class group $\mathbf{Mod}(G)$.

Consider a Thurston map $f: (S^2, A) \hookrightarrow$. A multicurve \mathcal{C} is *f-invariant* if $\mathcal{C} = f^{-1}(\mathcal{C})$, up to isotopy, identifying parallel curves and removing inessential curves. The Thurston matrix $T_{f, \mathcal{C}}$ is then an endomorphism of $\mathbb{Q}\mathcal{C}$.

The importance of invariant multicurves was made clear by a fundamental result of Thurston characterizing Thurston maps that are combinatorially equivalent to rational maps. An *annular obstruction* for a Thurston map f is an f -invariant multicurve \mathcal{C} such that the spectral radius of $T_{f, \mathcal{C}}$ is ≥ 1 .

Theorem 3.5 (Thurston [13]). *Let $f: (S^2, P_f) \hookrightarrow$ be a Thurston map with hyperbolic orbifold, see §7. Then f is combinatorially equivalent to a rational map if and only if f admits no annular obstruction. Furthermore, in that case the rational map is unique up to conjugation by Möbius transformations.*

3.3. Decomposition along multicurves. Consider again a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ with multicurves satisfying as before $\mathcal{D} \subseteq f^{-1}(\mathcal{C})$. By Theorem 3.4, there are sphere tree of groups decompositions \mathfrak{X} of $G = \pi_1(S^2 \setminus A, *)$ and \mathfrak{Y} of

$H = \pi_1(S^2 \setminus C, \dagger)$. Using the van Kampen Theorem for bisets, we shall decompose the H - G -biset $B(f)$ into an \mathfrak{Y} - \mathfrak{X} -tree of bisets $\mathfrak{B}(f)$.

Up to homotopy, the map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ may be written as a correspondence

$$(9) \quad (S^2, C, \mathcal{D}) \xleftarrow{i} (S^2, f^{-1}(A), f^{-1}(\mathcal{C})) \xrightarrow{f} (S^2, A, \mathcal{C});$$

the map f is the same as the original map f , but is now a covering, and we specify the map i as follows. It first forgets all points in $f^{-1}(A) \setminus C$ and all curves in $f^{-1}(\mathcal{C})$ that are not isotopic to curves in \mathcal{D} . It squeezes all annuli between the remaining curves in $f^{-1}(\mathcal{C})$ that are isotopic in $S^2 \setminus C$, and maps them to the corresponding curve in \mathcal{D} . This defines uniquely i as a monotone map (preimages of connected sets are connected), up to isotopy.

Let us say that a curve $\gamma \in f^{-1}(\mathcal{C})$ is

essential if γ is homotopic rel C to a curve in \mathcal{D} ,
non-essential otherwise.

Lemma 3.6. *Consider $\gamma \in f^{-1}(\mathcal{C})$. If both connected components of $S^2 \setminus \gamma$ contain essential curves of $f^{-1}(\mathcal{C})$, then γ is non-trivial rel C .*

Proof. If γ is a non-essential curve, then it surrounds a disc homotopic rel C to a point in S^2 . Therefore, all curves in this disc are peripheral or trivial rel C . \square

The respective multicurves \mathcal{D} , $f^{-1}(\mathcal{C})$ and \mathcal{C} in (9) define 1-dimensional covers as in Definition 3.1; following [3, §4.2] we define the tree of bisets associated with f as follows:

Definition 3.7 (Sphere tree of bisets $\mathfrak{B}(f)$). Consider a map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ and decompose it as in (9). Let \mathfrak{Y} and \mathfrak{X} be the sphere trees of groups associated with (S^2, C, \mathcal{D}) and (S^2, A, \mathcal{C}) respectively as in Definition 3.1. Then the \mathfrak{Y} - \mathfrak{X} sphere tree of bisets $\mathfrak{B} = \mathfrak{B}(f)$ is constructed as follows.

Vertices of \mathfrak{B} are in bijection with components of $f^{-1}(\mathcal{C})$ and closures of components of $S^2 \setminus f^{-1}(\mathcal{C})$; call the former *curve vertices* and the latter *sphere vertices*. Consider a vertex z of $\mathfrak{B}(f)$; it is represented by a subset S_z of S^2 , and is either a curve or a small sphere. The graph morphism $\rho: \mathfrak{B} \rightarrow \mathfrak{X}$ is given by the covering f so that $f(S_z) = S_{\rho(z)}$.

Define the map $\lambda: \mathfrak{B} \rightarrow \mathfrak{Y}$ such that $i(S_z) \subset S_{\lambda(z)}$ for every object $z \in \mathfrak{B}$ and such that

- (TB₁) λ is a monotone map; i.e. $\lambda^{-1}(z)$ is a subtree of \mathfrak{B} for every $z \in \mathfrak{B}$; and
- (TB₂) λ maps essential curve vertices to essential curve vertices.

For every $z \in \mathfrak{B}$ the $G_{\lambda(z)}$ - $G_{\rho(z)}$ -biset attached to z is

$$(10) \quad B(i: S_z \setminus f^{-1}(A) \rightarrow S_{\lambda(z)} \setminus C)^\vee \otimes B(f: S_z \setminus f^{-1}(A) \rightarrow S_{\rho(z)} \setminus A). \quad \triangle$$

By [3, Theorem 4.8], the fundamental biset of $\mathfrak{B}(f)$ is isomorphic to $B(f)$.

In the next Lemma 4.1 we will show that Conditions (TB₁) and (TB₂) uniquely specify $\lambda: \mathfrak{B} \rightarrow \mathfrak{Y}$; in particular these conditions are realizable. Let us also note that without these conditions there are in general many choices of λ , and therefore many slightly different graphs of bisets as in [3, §4.2] decomposing B .

Lemma 3.8 (Uniqueness of \mathfrak{B}). *Definition 3.7 determines the map $\lambda: \mathfrak{B} \rightarrow \mathfrak{Y}$ uniquely. Conditions (TB₁) and (TB₂) are equivalent to*

(TB') for every curve vertex $y \in \mathfrak{Y}$ and every vertex $z \in \mathfrak{B}$ we have $\lambda(z) = y$ if and only if $i(S_z) \subset S_y$.

Up to congruence of trees of bisets (see [3, Definition 3.29]), the sphere tree of bisets $\mathfrak{B} = \mathfrak{B}(f)$ is determined by the isotopy type of $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$.

Furthermore, for every curve vertex $e \in \mathfrak{Y}$ we have $i^{-1}(S_e) = \bigcup_{z \in \lambda^{-1}(y)} S_z$ and this set is an annulus (possibly a closed curve) bounded by two (possibly equal) curves in $f^{-1}(\mathcal{C})$ that are isotopic rel C to S_e .

Proof. Clearly, Condition (TB') implies the uniqueness of λ . Let us prove that Conditions (TB₁) and (TB₂) are equivalent to (TB'). This last property implies the monotonicity of λ (by monotonicity of i) and, clearly, λ maps essential curve vertices to curve vertices. Conversely, for every curve vertex $y \in \mathfrak{Y}$ the set $i^{-1}(S_y)$ is an annulus (possibly a closed curve) bounded by two curves $S_z, S_{z'} \in f^{-1}(\mathcal{C})$ isotopic rel C (possibly with $S_z = S_{z'}$). By Condition (TB₂) we have $\lambda(z) = \lambda(z')$, and by monotonicity of λ we have $i^{-1}(S_y) \subset \bigcup_{z \in \lambda^{-1}(y)} S_z$. Therefore, $i^{-1}(S_y) = \bigcup_{z \in \lambda^{-1}(y)} S_z$ because $i(S_v) \subset S_{\lambda(v)}$ for all vertices $v \in \mathfrak{B}$, and the claim of the lemma follows. The second claim follows from [3, Lemma 4.9] because λ, ρ are uniquely specified. The last claim has already been verified. \square

Note here one of the reasons we considered in Definition 3.1 the barycentric subdivision of the simpler decomposition with one vertex per small sphere and one edge per curve: our definition of graphs does not allow vertices to be mapped to midpoints of edges, as we would like to do in case z is a small sphere vertex and $i(S_z)$ is homotopic to a curve. Without taking a barycentric subdivision, we would have been forced to choose one of its two neighbouring spheres to which to map z ; and no such choice can be made canonical.

Theorem 3.9. *There is an algorithm that, given a sphere biset ${}_H B_G$ and algebraic multicurves \mathcal{C} in G and \mathcal{D} in H with \mathcal{D} contained in the B -lift of \mathcal{C} , computes the decomposition of B as a sphere tree of bisets.*

Proof. By Theorem 2.8 we may assume that ${}_H B_G$ is the biset of a sphere map $f: (S^2, C) \rightarrow (S^2, A)$. Then \mathcal{C} and \mathcal{D} define multicurves, still denoted by \mathcal{C} and \mathcal{D} , in (S^2, A) and (S^2, C) .

Choose $b \in B$ and decompose ${}_H B_G = HbG_b \otimes_{G_b} G_G$ as in (6). Recall from Lemmas 2.9 and 2.10 that this decomposition is the algebraic counterpart of (3) and that G_b is identified with the fundamental group of $(S^2, f^{-1}(A))$.

Denote by $(\Gamma_a)_{a \in A}$ the peripheral conjugacy classes in G , by $(\Delta_c)_{c \in C}$ the peripheral conjugacy classes in H , and by $(\Xi_a)_{a \in f^{-1}(A)}$ the peripheral conjugacy classes in G_b . Note that $(\Xi_a)_{a \in f^{-1}(A)}$ is easily computable as the set of all lifts of $(\Gamma_a)_{a \in A}$ through ${}_{G_b} G_G$, see Lemma 2.9.

Denote by $f^{-1}(\mathcal{C})$ the set of all lifts of \mathcal{C} through ${}_{G_b} G_G$. Again $f^{-1}(\mathcal{C})$ is easily computable and is the algebraic counterpart of the multicurve $f^{-1}(\mathcal{C})$ in $(S^2, f^{-1}(A))$. By Theorem 3.4, the sphere groups G, G_b, H can be decomposed relatively to $\mathcal{C}, f^{-1}(\mathcal{C}), \mathcal{D}$ as trees of groups $\mathfrak{X}, \mathfrak{K}, \mathfrak{Y}$ respectively.

By definition, the underlying tree of bisets decomposing B is the graph \mathfrak{K} . The map $\rho: \mathfrak{K} \rightarrow \mathfrak{X}$ is induced by inclusion: consider a vertex $z \in \mathfrak{K}$, with corresponding group G_z . Then $\rho(z)$ is the unique vertex of \mathfrak{X} such that $G_z \subseteq G_{\rho(z)}$.

By lifting curves in $f^{-1}(\mathcal{C})$ through HbG_b we compute the set of essential curve vertices in \mathfrak{K} as well the λ on this set. By monotonicity of λ (see the second claim

in Lemma 3.8) it has a unique, and thus computable, extension on the whole graph \mathfrak{K} .

As a graph, \mathfrak{B} is the underlying graph of \mathfrak{K} . Consider now a vertex $z \in \mathfrak{K}$; we define a $H_{\lambda(z)}\text{-}G_{\rho(z)}$ -biset B_z , to be put at z in the tree of bisets, as follows. We compute $bG_z = L_z b$ for a finitely generated subgroup L_z of H ; this is possible since elements of G_z stabilize b . We find (e.g. by enumeration) an element $h \in H$ with $L_z \leq H_{\lambda(z)}^h$. This is possible because $i(S_z) \subset S_{\lambda(z)}$ in the notation of (10). We finally set $B_z := H_{\lambda(z)} h b G_{\rho(z)}$, the subbiset of B given in (10). \square

3.4. Kameyama's theorem extended to spheres with multicurves. We are ready to extend Kameyama's algebraic characterization of combinatorial equivalence, taking multicurves into account. This will lead to algorithmic constructibility of sphere decompositions.

Theorem 3.10. *Let $f, g: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ be two sphere maps and let \mathfrak{Y} and \mathfrak{X} be the sphere trees of groups associated with (S^2, C, \mathcal{D}) and (S^2, A, \mathcal{C}) respectively as in Definition 3.1. Then f is isotopic to g by an isotopy mapping \mathcal{D} to \mathcal{C} if and only if the sphere trees of bisets ${}_{\mathfrak{Y}}\mathfrak{B}(f)_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{B}(g)_{\mathfrak{X}}$ are isomorphic.*

Proof. Since the sphere tree of bisets $\mathfrak{B}(f)$ is constructed out of $B(f)$ and \mathcal{C}, \mathcal{D} viewed as conjugacy classes in $\pi_1(S^2, C)$ and $\pi_1(S^2, A)$; and similarly $\mathfrak{B}(g)$ is constructed out of $B(g)$ and \mathcal{C}, \mathcal{D} , the trees of bisets $\mathfrak{B}(f)$ and $\mathfrak{B}(g)$ are isomorphic if and only if $B(f)$ and $B(g)$ are isomorphic by an isomorphism preserving the dynamics on multicurves.

By Theorem 2.8, this happens if and only if f is isotopic to g by an isotopy preserving \mathcal{D} and \mathcal{C} . \square

We recall that two graphs of bisets ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{C}_{\mathfrak{Y}}$ are called *conjugate* if there exists a biprincipal (see [3, §3.8]) graph of bisets ${}_{\mathfrak{X}}\mathfrak{I}_{\mathfrak{Y}}$ such that $\mathfrak{B} \otimes_{\mathfrak{X}} \mathfrak{I}$ and $\mathfrak{I} \otimes_{\mathfrak{Y}} \mathfrak{C}$ are isomorphic, namely have same underlying graphs and isomorphic vertex and edge bisets. If ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{Y}}\mathfrak{C}_{\mathfrak{Y}}$ are sphere trees of bisets, then ${}_{\mathfrak{X}}\mathfrak{I}_{\mathfrak{Y}}$ is also required to be a sphere tree of bisets.

Two Thurston maps $f_0: (S^2, A_0, \mathcal{C}_0) \hookrightarrow$ and $f_1: (S^2, A_1, \mathcal{C}_1) \hookrightarrow$ with respective invariant multicurves $\mathcal{C}_0, \mathcal{C}_1$ are *combinatorially equivalent* if there exists a path of Thurston maps $(f_t: (S^2, A_t, \mathcal{C}_t) \hookrightarrow)_{t \in [0,1]}$ connecting them.

Recall finally from [3, Definition 3.29] that two graphs of bisets ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ and ${}_{\mathfrak{X}'}\mathfrak{C}_{\mathfrak{X}'}$ are called *conjugate* if ${}_{\mathfrak{X}}\mathfrak{B}_{\mathfrak{X}}$ is isomorphic to $\mathfrak{I} \otimes_{\mathfrak{X}'} \mathfrak{C}_{\mathfrak{X}'} \otimes \mathfrak{I}^{\vee}$ for a biprincipal graph of bisets ${}_{\mathfrak{X}}\mathfrak{I}_{\mathfrak{X}'}$.

Corollary 3.11. *Let $f: (S^2, A, \mathcal{C}) \hookrightarrow$ and $g: (S^2, C, \mathcal{D}) \hookrightarrow$ be Thurston maps with respective invariant multicurves \mathcal{C}, \mathcal{D} . Then f, g are combinatorially equivalent if and only if the sphere trees of bisets of f, g are conjugate.*

Proof. As in the case of Thurston maps without multicurve (see Lemma 2.12), a combinatorial equivalence between Thurston maps with multicurves factors as a composition of an isotopy and a conjugation. The corollary therefore follows from Theorem 3.10. \square

4. RENORMALIZATION

Let us consider a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$. In this section we assume $\mathcal{D} = f^{-1}(\mathcal{C}) \text{ rel } C$. In this case \mathcal{D} naturally cuts f into exactly $\#\mathcal{D} + 1$ sphere bisets which we call *small bisets of the decomposition*.

It easily follows from the assumption $\mathcal{D} = f^{-1}(\mathcal{C}) \text{ rel } C$ that every non-essential curve $\gamma \in f^{-1}(\mathcal{C})$ is either trivial or peripheral. Since \mathfrak{B} is a tree, every edge vertex $z \in \mathfrak{B}$ disconnects \mathfrak{B} into exactly two connected components.

Lemma 4.1. *Let $z \in \mathfrak{B}$ be a curve vertex. Then z is essential if and only if every connected component of the graph $\mathfrak{B} \setminus \{z\}$ contains an object v with $\lambda(v) \neq \lambda(z)$.*

Proof. If z is a non-essential curve vertex, then S_z surrounds a disc D homotopic rel C to a point. Therefore, all objects representing small spheres and curves within D have the same image under λ as z . Conversely, suppose z is an essential curve vertex. Then $A_z := i^{-1}(S_{\lambda(z)}) = \bigcup_{v \in \lambda^{-1}(\lambda(z))} S_v$ is an annulus (possibly a closed curve) such that its boundary components are homotopic to S_z . Therefore, there are $S_v, S_w \subset S^2 \setminus A_z$ in different components of $S^2 \setminus S_z$ such that $\lambda(v) \neq \lambda(z) \neq \lambda(w)$. \square

Corollary 4.2. *Suppose that the underlying graph of \mathfrak{B} is not a singleton. Let $y \in \mathfrak{Y}$ be a sphere vertex. Then $\lambda^{-1}(y)$ contains a unique $z \in \mathfrak{B}$ that neighbours at least one essential curve vertex.*

Proof. Clearly there is at least one sphere vertex in $\lambda^{-1}(y)$ neighbouring at least one essential curve vertex. If there are two such sphere vertices $v, w \in \lambda^{-1}(y)$, then they are separated by a non-essential curve vertex; this contradicts Lemma 3.6. \square

4.1. Essential spheres. We return to a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$, keeping the convention $\mathcal{D} = f^{-1}(\mathcal{C}) \text{ rel } C$, and consider its decomposition as tree of bisets. The curve vertices in $\mathfrak{B}(f)$ correspond to edges in the decomposition of $(S^2, f^{-1}(A), f^{-1}(\mathcal{C}))$, and to connected components of $f^{-1}(\mathcal{C})$, while the sphere vertices $z \in \mathfrak{B}(f)$ correspond to small spheres S_z . We classify sphere vertices in $\mathfrak{B}(f)$ as follows: a sphere vertex $z \in \mathfrak{B}(f)$ is

- trivial** if S_z is homotopic rel C to a point in (S^2, C) ;
- annular** if S_z is homotopic rel C to a curve in \mathcal{D} ;
- essential** otherwise.

A trivial vertex is thus a dynamically-irrelevant small sphere in $S^2 \setminus f^{-1}(\mathcal{C})$: it gets blown down to a point (marked or not) by projecting to (S^2, C) . We have the following characterization in terms of graph combinatorics:

Lemma 4.3. *Suppose that the underlying tree of \mathfrak{B} is not a singleton. A sphere vertex $z \in \mathfrak{B}$ is trivial if there is a non-essential curve vertex in \mathfrak{B} separating z from at least one essential curve vertex; it is annular if it separates two essential curve vertices of \mathfrak{B} mapping to the same curve vertex of \mathfrak{Y} ; and it is a sphere vertex if $\lambda(z)$ is a sphere vertex and z is adjacent to an essential curve vertex.*

Proof. A sphere vertex $z \in \mathfrak{B}$ is trivial if and only if S_z lies within a disc that is contractible rel C and is surrounded by a non-essential curve in $f^{-1}(\mathcal{C})$; this is equivalent to the condition stated in the lemma.

If a sphere vertex $z \in \mathfrak{B}$ is annular, then there are two components in ∂S_z homotopic rel C to a curve in \mathcal{D} ; namely S_z separates two homotopic rel C essential curves in $f^{-1}(\mathcal{C})$; this is equivalent to the condition stated in the lemma. Conversely, if S_z separates two essential curves in $f^{-1}(\mathcal{C})$ that are homotopic rel C , then S_z is itself homotopic rel C to these curves.

By definition, if $z \in \mathfrak{B}$ is an essential sphere vertex, then $\lambda(z)$ is a sphere vertex. Conversely, if $z \in \mathfrak{B}$ is a sphere vertex while $\lambda(z)$ is a curve vertex, then z is either

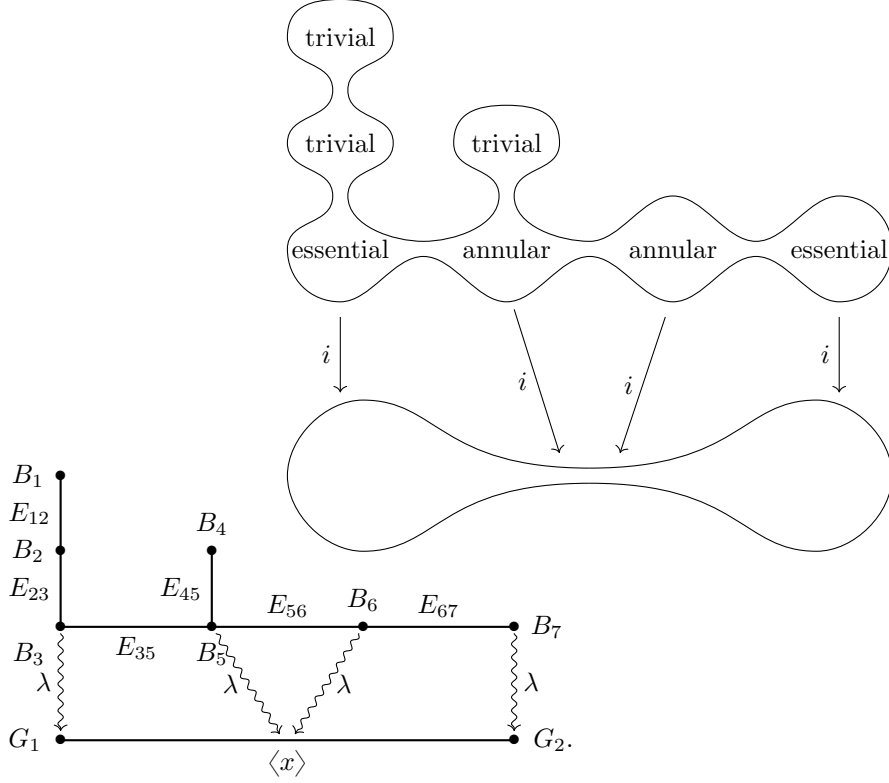


FIGURE 1. Trivial, annular and essential small spheres, and the corresponding tree of bisets (note that the actual graph of bisets is a barycentric subdivision of the one shown on the picture)

a trivial or an annular sphere vertex. By Corollary 4.2, for a vertex $y \in \mathfrak{Y}$ there is a unique vertex $z' \in \lambda^{-1}(y)$ that neighbours an essential curve vertex of \mathfrak{B} . On one hand, all sphere vertices in $\lambda^{-1}(y) \setminus \{z'\}$ are trivial (by the first claim of this lemma), on the other hand z' is neither trivial nor annular (by the first two claims of this lemma); i.e. z' is an essential sphere vertex. \square

Corollary 4.4. *There is a computable bijection between the small spheres in $S^2 \setminus \mathscr{D}$ and the essential spheres in $S^2 \setminus f^{-1}(\mathscr{C})$.*

Proof. This follows from Lemma 4.3 and Corollary 4.2. All conditions are expressed in terms of the map of finite graphs $\lambda: \mathfrak{B} \rightarrow \mathfrak{Y}$, so are computable by Theorem 3.9. \square

Let us remark that the classification of small spheres in $(S^2, f^{-1}(A), f^{-1}(\mathscr{C}))$ depends only on the forgetful map $(S^2, f^{-1}(A), f^{-1}(\mathscr{C})) \xrightarrow{\mathbb{1}} (S^2, C, \mathscr{D})$; see Figure 1.

4.2. Epic-monic factorizations. Recall from [3, Lemma 2.7] that for a transitive H - G -biset B there exist a group K and homomorphisms $\phi: K \rightarrow G$, $\psi: K \rightarrow H$ such that $B = B_\psi^\vee \otimes_K B_\phi$. Moreover, there is a minimal such K .

Every group homomorphism factors as a composition of a surjective (“epic”) morphism with an injective (“monic”) morphism. Let us apply this factorization to ϕ and ψ ; that is, set $P := \psi(K)$, $Q := \phi(K)$, and view ψ and ϕ as homomorphisms onto P and Q respectively, followed by natural inclusions. We get the factorization

$$B = {}_H H_P \otimes B_\psi^\vee \otimes_K B_\phi \otimes_Q G_G.$$

Combining the last three terms into a biset ${}_P B'_G$ we get the *left epic-monic* factorization

$$B = {}_H H_P \otimes_P B'_G.$$

Below is a direct way how to calculate P and B' ; its proof is immediate.

Lemma 4.5. *Let B be a transitive H - G -biset. Choose $b \in B$ and let P be the stabilizer of $b \otimes \{\cdot\}$ in $B \otimes_G \{\cdot\}$. Then $B' = PbG$ and $B = {}_H H_P \otimes PbG$. \square*

We remark that the left epic-monic decomposition of a sphere biset ${}_H B_G$ is trivial (i.e. $P = H$) because the G -action is transitive. Let us now generalize Lemma 4.5 to the context of graphs of bisets. Recall from [3] that for bisets ${}_H B_G, {}_{H'} B'_{G'}$ an *intertwiner* is a triple of maps $(\gamma: H \rightarrow H', \beta: B \rightarrow B', \alpha: G \rightarrow G')$ satisfying $\beta(hbg) = \gamma(h)\beta(b)\alpha(g)$ for all $h \in H, b \in B, g \in G$.

Proposition 4.6. *Let $\mathfrak{Y}\mathfrak{B}_{\mathfrak{X}}$ be a graph of bisets and assume that every B_z is a transitive $G_{\lambda(z)}$ - $G_{\rho(z)}$ -biset. Let \mathfrak{X}° be the graph of groups obtained from \mathfrak{X} by replacing each group G_z with the trivial group. Denote by $(\mathfrak{X}, \mathfrak{X}^\circ)$ the \mathfrak{X} - \mathfrak{X}° -graph of bisets with $\lambda = \rho = 1$ and bisets $B_z = G_z \{\cdot\}_1$ for all $z \in (\mathfrak{X}, \mathfrak{X}^\circ)$.*

Consider the graph of bisets $\mathfrak{B} \otimes (\mathfrak{X}, \mathfrak{X}^\circ)$. For every $z \in \mathfrak{B}$, choose $b_z \in B_z$. Let $P_{\lambda(z)} \leq G_{\lambda(z)}$ be the stabilizer of $b_z \otimes \{\cdot\}$ in $B_{(z, \rho(z))}$, with $(z, \rho(z))$ viewed as an object in $\mathfrak{B} \otimes (\mathfrak{X}, \mathfrak{X}^\circ)$. As in Lemma 4.5, decompose $B_z = G_{\lambda(z)} \otimes_{P_{\lambda(z)}} B'_z$.

Let \mathfrak{P} be the graph of groups with underlying graph \mathfrak{B} and group P_z at every $z \in \mathfrak{P}$, and with morphisms $P_z \rightarrow P_{z^-}$ given by $g \rightarrow (g^-)^{\gamma_z}$ with γ_z chosen so that $\gamma_z^{-1}(b_{(z, \rho(z))})^- = b_{(z^-, \rho(z^-))}$.

Let \mathfrak{B}'_z be the \mathfrak{P} - \mathfrak{X} -graph of bisets with biset $P_z(B'_z)_{G_{\rho(z)}}$ attached to each $z \in \mathfrak{B}'$ and biset intertwiners $B'_z \rightarrow B'_{z^-}$ given by $b \rightarrow \gamma_z^{-1}b^-$. Let $(\mathfrak{Y}, \mathfrak{P})$ be the \mathfrak{Y} - \mathfrak{P} -graph of bisets with underlying graph \mathfrak{B} and with biset $B_z = G_{\lambda(z)} G_{\lambda(z)P_z}$ attached to $z \in (\mathfrak{Y}, \mathfrak{P})$ with biset intertwiners $B_z \rightarrow B_{z^-}$ given by $g \rightarrow g\gamma_z$.

Then \mathfrak{B} decomposes as

$$(11) \quad \mathfrak{B} = (\mathfrak{Y}, \mathfrak{P}) \otimes \mathfrak{B}'.$$

Proof. Follows directly from the construction and the definition of the tensor product: $(\mathfrak{Y}, \mathfrak{P}) \otimes \mathfrak{B}'$ is \mathfrak{B} as a graph with $G_{\lambda(z)} G_{\lambda(z)P_z} \otimes B'_z = B_z$ attached to $z \in (\mathfrak{Y}, \mathfrak{P})$. \square

4.3. Small sphere maps. The classification of small spheres can be directly seen at the algebraic level. Consider a sphere biset ${}_H B_G$, multicurves \mathcal{C}, \mathcal{D} in G, H respectively such that \mathcal{D} is contained in the B -lift of \mathcal{C} . Let $\mathfrak{X}, \mathfrak{Y}$ denote the tree of groups decompositions of G, H along \mathcal{C}, \mathcal{D} respectively, and consider the corresponding tree of bisets decomposition $\mathfrak{Y}\mathfrak{B}_{\mathfrak{X}}$ of B .

We express algebraically the notions of trivial, annular and essential sphere vertices using the epic-monic factorization from §4.2:

Definition 4.7 (Algebraically trivial, annular, essential sphere vertices). Let $\mathfrak{Y}\mathfrak{B}_{\mathfrak{X}}$ be a sphere tree of bisets. A sphere vertex of $z \in \mathfrak{B}_{\mathfrak{X}}$ is

trivial if B_z is of the form $G_{\lambda(z)} \otimes_{P_z} B'_z$ for a sphere P_z - $G_{\rho(z)}$ -biset B'_z and a subgroup $P_z \leq G_{\lambda(z)}$ generated by a representative of a peripheral in $\pi_1(\mathfrak{Y})$ or trivial conjugacy class, so that the sphere structure of P_z is viewed accordingly as either that of $\pi_1(S^2)$ or as that of $\pi_1(S^2 \setminus \{0, \infty\})$;
annular if B_z is of the form $G_{\lambda(z)} \otimes_{P_z} B'_z$ for a sphere P_z - $G_{\rho(z)}$ -biset B'_z and a subgroup $P_z \leq G_{\lambda(z)}$ generated by a representative of a class in \mathcal{D} , so that the sphere structure of P_z is viewed as that of $\pi_1(S^2 \setminus \{0, \infty\})$;
essential if B_z is a sphere biset.

Similarly, a curve vertex $e \in \mathfrak{B}$ is

non-essential if B_e is of the form $G_{\lambda(e)} \otimes_{P_e} B'_e$ for a sphere P_e - $G_{\rho(e)}$ -biset B'_e and a subgroup $P_e \leq G_{\lambda(e)}$ generated by a representative of a conjugacy class that is peripheral or trivial in $\pi_1(\mathfrak{Y})$, so that the sphere structure of P_e is viewed accordingly as that of $\pi_1(S^2)$ or as that of $\pi_1(S^2 \setminus \{0, \infty\})$;
essential if B_e is a sphere biset. \triangle

Observe that all these cases are exclusive; for example, in the first, second and fourth cases the biset B_z is not a sphere biset.

Lemma 4.8. *Let $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ be a sphere map with associated tree of bisets $\mathfrak{B}(f)$. Then a small sphere vertex of $\mathfrak{B}(f)$ is trivial, annular or essential if and only if the corresponding small sphere is respectively trivial, annular or essential, and a curve vertex of $\mathfrak{B}(f)$ is essential or non-essential if and only if the corresponding curve is respectively essential or non-essential.*

Proof. If $z \in \mathfrak{B}$ is a trivial sphere vertex, then $P_z := i_*\pi_1(S_z \setminus f^{-1}(A))$ is either trivial or generated by a representative of a peripheral conjugacy class. Decompose $i: S_z \rightarrow S_{\lambda(z)}$ as $S_z \xrightarrow{i} i(S_z) \hookrightarrow S_{\lambda(z)}$; so $B(i: S_z \rightarrow S_{\lambda(z)})^\vee$ factors as $G_{\lambda(z)} \otimes_{P_z} B''_z$ for a sphere biset B''_z . By (10), B_z is of the form $G_{\lambda(z)} \otimes_{P_z} B'_z$ for a sphere biset B'_z . If $z \in \mathfrak{B}$ is an annular sphere vertex, then $P_z := i_*\pi_1(S_z \setminus f^{-1}(A))$ is generated by a representative of a conjugacy class in \mathcal{D} . As above, B_z is of the form $G_{\lambda(z)} \otimes_P B'_z$.

Suppose that $z \in \mathfrak{B}$ is an essential sphere vertex. Observe $i_*\pi_1(S_z \setminus f^{-1}(A)) = \pi_1(S_{\lambda(z)} \setminus C)$ because $i^{-1}(S_{\lambda(z)}) \setminus S_z$ consists of finitely many contractible rel C discs. Since $G_{\lambda(z)}$ has at least three peripheral conjugacy classes, B_z does not factor as $G_{\lambda(z)} \otimes_P B'$ for a subgroup $P \leq G_{\lambda(z)}$ and it follows that B_z is a sphere biset.

The characterization of essential curve vertices is verified similarly. \square

Let $\mathcal{S}, \mathcal{T}, \mathcal{U}$ denote the collection of small spheres in the decomposition of $S^2 \setminus \mathcal{C}, S^2 \setminus f^{-1}(\mathcal{C}), S^2 \setminus \mathcal{D}$ respectively. By Corollary 4.4, there is a well-defined map, still written $f: \mathcal{U} \rightarrow \mathcal{S}$, sending each small sphere $U_w \subset S^2 \setminus \mathcal{D}$ to the image by f of the unique essential sphere of $S^2 \setminus f^{-1}(\mathcal{C})$ contained in U_w . To avoid cumbersome indices, we write interchangeably $f(U_w) = S_v$ and $f(w) = v$, defining by the latter a map from the vertex set of \mathfrak{Y} to the vertex set of \mathfrak{X} .

Lemma-Definition 4.9. *The induced map $\widehat{f}: \widehat{U}_w \rightarrow \widehat{S_{f(w)}}$ is a sphere map, called a small sphere map of f .*

Proof. Let T_z be the essential sphere above U_w . By construction, $i: T_z \rightarrow U_w$ is a homeomorphism onto its image, and components of $U_w \setminus i(T_z)$ are contractible.

The space \widehat{U}_w can equivalently be constructed by attaching a copy of the unit disk \mathbb{D} , along its boundary, to every S^1 -boundary component. The disk is marked

at its centre. In this manner, U_w is seen as a subset of \widehat{U}_w . We perform the same construction on T_z and on $S_{f(w)}$.

The map $f: T_z \rightarrow S_{f(w)}$ extends naturally to a branched covering $g: \widehat{T}_z \rightarrow \widehat{S}_{f(w)}$ with branched values in $A \cup \{\text{centres of disks}\}$: on T_z it is defined as f , while on a disk \mathbb{D} with boundary curve C mapping by degree d to $f(C)$ it is defined as $z^d: \mathbb{D} \rightarrow \mathbb{D}$.

The map $i: T_z \rightarrow U_w$ extends naturally to a homeomorphism $j: \widehat{T}_z \rightarrow \widehat{U}_w$: on T_z it is defined as i , while on disks \mathbb{D} it is defined as the identity. We set $\hat{f} = j^{-1}g$ and note that it is a branched covering. \square

These definitions admit direct analogues in the algebraic setting of a sphere tree of bisets ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{X}}$. By Lemma 4.3, for every vertex $w \in \mathfrak{Y}$ there is a unique essential vertex $z \in \mathfrak{B}$ in $\lambda^{-1}(w)$, and $\rho(z) = v$ for some sphere vertex $v \in \mathfrak{X}$. We define a map B_* from the vertex set of \mathfrak{Y} to the vertex set of \mathfrak{X} by

$$B_*(w) := v \text{ if } \lambda(z) = w \text{ and } \rho(z) = v \text{ and } z \text{ essential.}$$

We may also extend B_* into a map from the geometric realization of \mathfrak{Y} to that of \mathfrak{X} . Recall that, for a graph \mathfrak{X} , its geometric realization \mathfrak{X}_0 is the topological space

$$\mathfrak{X}_0 = \mathfrak{X} \times [0, 1] / \{(x, t) = (\bar{x}, 1 - t), (x^-, t) = (x, 0) \forall x \in \mathfrak{X}, \forall t \in [0, 1]\}.$$

The map

$$\lambda^{-1}: \{\text{sphere vertices of } \mathfrak{Y}\} \rightarrow \{\text{essential sphere vertices of } \mathfrak{B}\}$$

extends into an essentially unique (up to isotopy rel sphere vertices) tree embedding $\lambda^{-1}: \mathfrak{Y}_0 \hookrightarrow \mathfrak{B}_0$. Indeed, any geodesic ℓ in \mathfrak{Y}_0 lifts under λ into a unique geodesic traveling through essential curve vertices, see Lemma 3.6. Composing with ρ , we obtain the map $B_*: \mathfrak{Y}_0 \rightarrow \mathfrak{X}_0$.

Example 4.10. The map B_* is closely related to the self-map of the “Hubbard tree” associated with a complex polynomial; we refer to [3, §5.2] and the original references [11, 12].

Consider a complex polynomial p and let $g := p \uplus \bar{p}$ be the formal mating of p with its complex conjugate. Let us denote by $p: \mathcal{T} \hookrightarrow$ the Hubbard tree of p ; the Hubbard tree $\bar{p}: \bar{\mathcal{T}} \hookrightarrow$ of \bar{p} is obtained from $p: \mathcal{T} \hookrightarrow$ by applying complex conjugation.

For every edge $e \in \mathcal{T}$ there is a simple closed curve γ_e intersecting \mathcal{T} once at e and intersecting $\bar{\mathcal{T}}$ once at \bar{e} such that γ_e follows external rays of p and \bar{p} away from e and \bar{e} . If the edge $e_i \in \mathcal{T}$ covers $d_{i,j}$ times $e_j \in \mathcal{T}$, then $g^{-1}(\gamma_{e_j})$ has exactly $d_{i,j}$ components isotopic to γ_{e_i} and each of these components is a degree-one preimage of γ_{e_j} . Therefore, $\mathcal{C}_{\mathcal{T}} := \{\gamma_e: e \in \mathcal{T}\}$ is an invariant multicurve with Thurston matrix $T_{g, \mathcal{C}_{\mathcal{T}}} = (d_{i,j})$, an integer matrix; all the small maps in this decomposition are finite-order or monomial rational maps, so $\mathcal{C}_{\mathcal{T}}$ is the canonical obstruction of g .

Let ${}_{\mathfrak{Y}}\mathfrak{B}_{\mathfrak{Y}}$ be the graph of bisets encoding the decomposition of g relative to $\mathcal{C}_{\mathcal{T}}$, and let ${}_{\mathfrak{X}}\mathfrak{T}_{\mathfrak{X}}$ be the graph of bisets associated with $p: \mathcal{T} \hookrightarrow$ as in [3, §5.2]. Sphere vertices of \mathfrak{Y} are in bijection with vertices of \mathcal{T} and with non-edge vertices of \mathfrak{X} . Curve vertices of \mathfrak{Y} are in bijection with edges of \mathcal{T} . Essential sphere vertices of \mathfrak{B} are in bijection with essential vertices of \mathcal{T}^1 .

We conclude that B_* , as a self-map of sphere vertices of \mathfrak{Y} , coincides with p , as a self-map of vertices of \mathcal{T} .

At the graph level $\lambda, \rho: \mathfrak{B} \rightarrow \mathfrak{Y}$ are the same as $\lambda, \rho: \mathfrak{T} \rightarrow \mathfrak{X}$. Furthermore, there is a semiconjugacy $\pi: \mathfrak{Y} \mathfrak{B} \mathfrak{Y} \rightarrow \mathfrak{X} \mathfrak{T} \mathfrak{X}$ given by identifying the northern and southern hemisphere. More precisely, consider a sphere vertex $y \in \mathfrak{Y}$. Its associated group G_y maps to the corresponding group $G_{\pi(y)} = \mathbb{Z}/\text{ord}(\pi(y))$ in \mathfrak{X} by sending the peripheral generator in the northern and southern hemisphere to 1 and -1 respectively and all other peripheral generators to 0.

4.4. Refinement of sphere trees of groups. Recall from [3, §3.2] that the fundamental group of a graph of groups does not change under the operations “split an edge” and “add an edge”. Recall also that, in a sphere tree of groups, a peripheral conjugacy class in a vertex group is vacant if it intersects no image of an edge group.

If \mathfrak{X} is a stable sphere tree of groups, then we adjust these operations to respect the sphere structure as follows:

- (1) **split a curve vertex:** choose a curve vertex $v \in \mathfrak{X}$. Let e_0, \bar{e}_0 and e_1, \bar{e}_1 be the pair of edges adjacent to v . Declare v to be a sphere vertex add split e_0, \bar{e}_0 and e_1, \bar{e}_1 as follows. Add a curve vertex ℓ_0 to \mathfrak{X} and replace e_0, \bar{e}_0 by $e_{00}, \bar{e}_{00}, e_{01}, \bar{e}_{01}$ with $e_{00}^- = e_0^-, e_{00}^+ = \ell_0 = e_{01}^-, e_{01}^+ = e_0^+$; and make a similar operation for the pair e_1, \bar{e}_1 : replace it by edges $e_{10}, \bar{e}_{10}, e_{11}, \bar{e}_{11}$ and a new curve vertex ℓ_1 . Declare all groups to be equal to G_v with the obvious maps between them;
- (2) **add an edge:** choose a vertex $v \in V$, and either trivial or a vacant peripheral class Γ of v . Let H be a (trivial or cyclic) subgroup of G_v generated by a representative in $h \in \Gamma$. Add a new sphere vertex w to \mathfrak{X} , add a new curve vertex ℓ to \mathfrak{X} , and add new edges e_0, \bar{e}_0 and e_1, \bar{e}_1 with $e_0^- = v, e_0^+ = \ell$ and $e_1^- = \ell, e_1^+ = w$. Define the new groups by $G_{e_0} = G_\ell = G_{e_1} = G_w = H$, with the obvious maps between them. The peripheral conjugacy classes of H form either empty set in case H is trivial or $\{h, h^{-1}\}$ in case H is cyclic.

A *refinement* of a graph of groups \mathfrak{X} is a graph of groups obtained from \mathfrak{X} by applying finitely many times the above operations (1) and (2). An *unstable sphere tree of groups* is a refinement of a stable sphere tree of groups.

Let \mathfrak{P} be a refinement of \mathfrak{X} . Define a graph map $\lambda: \mathfrak{P} \rightarrow \mathfrak{X}$ by sending vertex $v \in \mathfrak{P}$ to the vertex of \mathfrak{X} from which v was refined. Define the tree of bisets $(\mathfrak{X}, \mathfrak{P})$ to be \mathfrak{P} as a graph with λ as above, $\rho := \mathbb{1}$, and with $B_z = {}_{G_{\lambda(z)}}G_{\lambda(z)}G_{\rho(z)}$ with natural maps between them. Clearly, the fundamental biset of $(\mathfrak{X}, \mathfrak{P})$ is isomorphic to ${}_{\pi_1(\mathfrak{X})}\pi_1(\mathfrak{X})_{\pi_1(\mathfrak{X})}$.

Lemma 4.11. *Let \mathfrak{P} be a sphere refinement of a sphere tree of groups \mathfrak{X} . Then for any vacant peripheral class Γ of \mathfrak{X} there is a unique vacant peripheral class of Γ' of \mathfrak{P} identified with Γ via $(\mathfrak{X}, \mathfrak{P})$.*

Proof. The tree of groups \mathfrak{P} is obtained from \mathfrak{X} via operations “split a curve vertex” and “add an edge”; these operations clearly respect the set of vacant peripheral classes in a sphere tree of groups. \square

4.5. Sphere trees of bisets. We give in this section a characterization of those trees of bisets that come from sphere maps with invariant multicurve. Combined with Theorem 3.10, this extends Theorem 2.8 to branched coverings with multicurves.

Definition 4.12. A tree of bisets $\mathfrak{Y} \mathfrak{B} \mathfrak{X}$ is a *sphere tree of bisets* if

- (ST₁) \mathfrak{Y} and \mathfrak{X} are stable sphere trees of groups;
 (ST₂) \mathfrak{B} is left-free, see [3, Definition 3.16], and B_z is transitive for all $z \in \mathfrak{B}$;
 write $\mathfrak{B} = (\mathfrak{Y}, \mathfrak{P}) \otimes \mathfrak{B}'$ as in (11);
 (ST₃) \mathfrak{P} has a sphere structure and is a sphere refinement of \mathfrak{Y} , every biset in \mathfrak{B}' is a sphere biset. \triangle

Lemma 4.13. *The tree of bisets $\mathfrak{B}(f)$ associated with a map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ as in Definition 3.7 with $\mathcal{D} = f^{-1}(\mathcal{C})$ rel C is a sphere tree of bisets.*

Proof. Condition (ST₁) is immediate. Condition (ST₂) follows from [3, Proposition 4.13] and Lemma 2.7. Condition (ST₃) follows from Lemma 4.8: for every $z \in \mathfrak{B}$ in the decomposition $B_z = G_{\lambda(z)} \otimes_{P_z} B'_z$ as in Lemma 4.5 the biset B'_z is sphere with the group P_z either trivial or generated by a peripheral conjugacy class in B_z . \square

Theorem 4.14. *The fundamental biset of a sphere tree of bisets $\mathfrak{Y}\mathfrak{B}\mathfrak{X}$ is a sphere biset.*

Proof. Let \mathfrak{B} be a sphere tree of bisets. We check that $\pi_1(\mathfrak{B})$ satisfies Definition 2.6.

By [3, Corollary 3.21], the biset $\pi_1(\mathfrak{B})$ is left free; and since each biset B_z is right transitive so is $\pi_1(\mathfrak{B})$. This is (SB₁).

Next we check (SB₃). Write $\mathfrak{B} = (\mathfrak{Y}, \mathfrak{P}) \otimes \mathfrak{B}'$ as in (11). By Lemma 4.11 the set of vacant conjugacy classes in \mathfrak{X} and in \mathfrak{P} are naturally identified. Let Γ be a vacant conjugacy class in \mathfrak{P} , say in P_z . Since B'_z is a sphere biset, Γ appears exactly once in the multiset of all lifts of all vacant peripheral conjugacy classes of $G_{\rho(z)}$.

We check (SB₃) by a counting argument. A cycle c as in (5) is *critical* of *multiplicity* $\text{length}(c) - 1$ if $\text{length}(c) > 1$. Let d be the degree (i.e. the number of left orbits) of $\pi_1(\mathfrak{B})$. We show that the number, counting multiplicity, of all critical cycles associated with vacant peripheral classes in \mathfrak{B} is $2d - 2$. Let n be the number of sphere vertices in \mathfrak{X} and m be the number of sphere vertices in \mathfrak{B} . Then the number of curve vertices in \mathfrak{X} and in \mathfrak{B} is $n - 1$ and $m - 1$ respectively. We also note $n \leq m < dn$.

Consider a sphere vertex $x \in \mathfrak{X}$. Then the number of critical cycles in $\bigsqcup_{z \in \rho^{-1}(x)} B_z$ that are associated with all peripheral classes of G_x is equal to $2(d - \#(\rho^{-1}(x)))$. By taking the sum over all sphere vertices $x \in \mathfrak{X}$ we see that the number of all critical cycles in \mathfrak{B} that are associated with all peripheral classes of all sphere groups in \mathfrak{X} is equal to $2(dn - m)$. From this number we will now subtract the number of whose critical cycle that are associated with non-vacant peripheral classes.

Consider a curve vertex $x \in \mathfrak{X}$; in two neighbouring sphere vertices, say v and v' , there are exactly two non-vacant peripheral conjugacy classes, say Γ and Γ' , defined by embedding G_x into G_v and into $G_{v'}$. The number of critical cycles in $\bigsqcup_{z \in \rho^{-1}(x)} B_z$ that are associated with Γ and Γ' is equal to $2(d - \#(\rho^{-1}(x)))$. Taking the sum over all curve vertices $x \in \mathfrak{X}$, we get $2(d(n - 1) - m + 1)$. Finally,

$$2(dn - m) - 2(d(n - 1) - m + 1) = 2d - 2,$$

which is (SB₃). \square

Corollary 4.15. *Let $\mathfrak{Y}\mathfrak{B}\mathfrak{X}$ be a sphere tree of bisets. Write \mathfrak{Y} as the sphere tree of groups associated with (S^2, C, \mathcal{D}) and associate similarly \mathfrak{X} with (S^2, A, \mathcal{C}) . Then*

there exists a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$, unique up to isotopy rel $C \cup \mathcal{D}$, whose graph of bisets is isomorphic to \mathfrak{B} .

Proof. Follows from Theorems 4.14, 2.8 and 3.10. \square

4.6. The dynamical situation. In the dynamical situation in which $A = C$ and $\mathcal{C} = \mathcal{D}$, we have a Thurston map $f: (S^2, A, \mathcal{C}) \hookrightarrow$ with an invariant multicurve \mathcal{C} . We then also have dynamics $f: \mathcal{S} \hookrightarrow$ on the set \mathcal{S} of small spheres of $S^2 \setminus \mathcal{C}$. We define the set of *return maps*, also called *small Thurston maps*,

$$R(f, A, \mathcal{C}) := \{f^e: S \hookrightarrow \mid S \in \mathcal{S}, f^e(S) = S \text{ with } e \text{ minimal}\}.$$

Similarly algebraically, in the dynamical situation in which $G = H$ and $\mathcal{C} = \mathcal{D}$, we have a biset ${}_G B_G$ with an invariant multicurve \mathcal{C} . Let us denote by ${}_x \mathfrak{B}_x$ the tree of bisets decomposition of B , and by V the set of sphere vertices of \mathfrak{X} , with essential biset B_v corresponding to $v \in V$, as given by Lemma 4.3. We define the set of *return bisets*, also called *small Thurston bisets*,

$$(12) \quad R(B, \mathcal{C}) := \{B_v \otimes B_{B_*(v)} \otimes \cdots \otimes B_{B_*^{e-1}(v)} \hookrightarrow \mid v \in V, B_*^e(v) = v \text{ with } e \text{ minimal}\},$$

namely the bisets obtained by following a cycle in the tree of bisets into which B decomposes. By Lemma 4.8 all bisets in $R(B, \mathcal{C})$ are sphere bisets.

4.7. Distinguished conjugacy classes. Let (S^2, A) be a marked sphere, and let \mathcal{C} be a multicurve. It is often convenient to treat similarly the conjugacy classes describing elements of A and of \mathcal{C} . Consider \mathfrak{X} a sphere tree of groups. The set of *distinguished conjugacy classes* X of \mathfrak{X} is the set of all peripheral conjugacy classes of all vertex groups with two conjugacy classes identified if they are related by an edge. Equivalently, if \mathfrak{X} is the tree of groups decomposition of (S^2, A, \mathcal{C}) , then X is in bijection with $A \cup \mathcal{C}$. Note that the set of geometric edges of \mathfrak{X} is naturally a subset of the distinguished conjugacy classes.

The following algorithm determines when a bijection between (possibly peripheral) multicurves is induced by a homeomorphism between the underlying spheres:

Algorithm 4.16. GIVEN \mathfrak{X} and \mathfrak{Y} two sphere trees of groups with distinguished conjugacy classes X and Y , and given a bijection $h: X \rightarrow Y$,
DECIDE whether $h: X \rightarrow Y$ promotes to a conjugator \mathfrak{J} from \mathfrak{X} to \mathfrak{Y} , and if so
CONSTRUCT \mathfrak{J} AS FOLLOWS:

- (1) Check whether h restricts to an isomorphism between the geometric edge sets of \mathfrak{X} and \mathfrak{Y} . If not return **fail**.
- (2) Check whether the isomorphism between the edge sets promotes into a graph-isomorphism $h: \mathfrak{X} \rightarrow \mathfrak{Y}$. If not, return **fail**.
- (3) For a sphere vertex $v \in \mathfrak{X}$ let $\Gamma_v \subset X$ be the set of peripheral conjugacy classes of G_v . Check whether $h(\Gamma_v) = \Gamma_{h(v)}$ for all vertices $v \in \mathfrak{X}$. If not, return **fail**.
- (4) For every sphere vertex $v \in \mathfrak{X}$ choose an isomorphism $\phi(v): G_v \rightarrow G_{h(v)}$ compatible with $h: \Gamma_v \rightarrow \Gamma_{h(v)}$. For every edge $e \in \mathfrak{Y}$ choose an isomorphism $\phi(e): G_e \rightarrow G_{h(e)}$.
- (5) Set $\mathfrak{J} := \mathfrak{X}$, $\lambda := 1$, $\rho := h$, $B_z := G_{h(z)}$ for all $z \in \mathfrak{J}$; the left action of G_z on B_z is via $\phi(z)$, the right action is natural, and the inclusion of B_e into

B_{e^-} is via $1 \mapsto g$, for any $g \in G_{h(z)}$ with $()^g \circ \phi(e) = \phi(e^-)$, if we identify G_e with a subgroup of G_{e^-} .

(6) Return \mathfrak{J} .

Let $\mathfrak{X}\mathfrak{B}_{\mathfrak{X}}$ and $\mathfrak{Y}\mathfrak{C}_{\mathfrak{Y}}$ be two sphere trees of bisets. If \mathfrak{J} be a conjugator between \mathfrak{X} and \mathfrak{Y} , then $\mathfrak{J} \otimes \mathfrak{C} \otimes \mathfrak{J}^\vee$ is an \mathfrak{X} -graph of bisets. Recall from §6.2 the notations $\mathbf{Mod}(\mathfrak{X})$ and $M(\mathfrak{B})$. The following two algorithms determine, given two trees of bisets that stabilize the same multicurve, whether they are twists of one another by mapping classes respecting the multicurve.

The first algorithm expresses, if possible, a sphere tree of bisets as a left multiple of another one. It relies on the following observation. Suppose that we want to construct a biprincipal sphere tree of bisets \mathfrak{T} over a sphere tree of groups \mathfrak{X} , and that its vertex and edge bisets are already given, so that only the intertwiners $T_e \rightarrow T_{e^-}$ need be specified at edges of \mathfrak{T} . Consider an edge pair $\{e, \bar{e}\}$. The bisets T_e and T_{e^\pm} may be identified with the groups G_e and G_{e^\pm} respectively; then the intertwiners $T_e \rightarrow T_{e^\pm}$ are defined by $1 \mapsto g_\pm$ for some $g_\pm \in G_{e^\pm}$ which commutes with the image of G_e . Since the G_{e^\pm} are free while G_e is Abelian, the element g_\pm may be chosen arbitrarily in $(G_e)^\pm$. All resulting choices of maps $T_e \rightarrow T_{e^\pm}$ are called *legal intertwiners*. In fact, writing $g_\pm = (h_\pm)^\pm$ for some $h_\pm \in G_e$, the isomorphism class of \mathfrak{T} depends only on $h_+(h_-)^{-1}$.

Algorithm 4.17. GIVEN $\mathfrak{X}\mathfrak{B}_{\mathfrak{X}}$ and $\mathfrak{X}\mathfrak{C}_{\mathfrak{X}}$ two sphere \mathfrak{X} -trees of bisets,
 DECIDE whether there is an $\mathfrak{M} \in \mathbf{Mod}(\mathfrak{X})$ such that $\mathfrak{C} \cong \mathfrak{M} \otimes \mathfrak{B}$, and if so CON-
 STRUCT \mathfrak{M} and the isomorphism AS FOLLOWS:

- (1) Try to construct an isomorphism of trees $h: \mathfrak{B} \rightarrow \mathfrak{C}$ mapping essential vertices into essential vertices such that $\lambda_{\mathfrak{B}}(z) = \lambda_{\mathfrak{C}} \circ h(z)$ and $\rho_{\mathfrak{B}}(z) = \rho_{\mathfrak{C}} \circ h(z)$ for every $z \in \mathfrak{B}$. If h does not exist, then return **fail**. Otherwise h is uniquely defined.
- (2) Choose an essential sphere vertex $v \in \mathfrak{B}$. Try to find $M_{\lambda(v)} \in \mathbf{Mod}(G_{\lambda(v)})$ such that $M_{\lambda(v)} \otimes B_v \cong C_{h(v)}$. If such $M_{\lambda(v)}$ do not exist, return **fail**. Otherwise set $S := \{v\}$ and run over pairs $\{e, \bar{e}\} \nsubseteq S$ but with $e^- \in S$ Steps 3–6.
- (3) If $\lambda(e) \notin \lambda(S)$, then do the following. (Note that in this case $\lambda(e)$ is an edge in \mathfrak{X} .) Add e and \bar{e} to S , let $M_{\lambda(e)}$ be a principal \mathbb{Z} -biset, choose any legal intertwiner $()^-: M_{\lambda(e)} \rightarrow M_{\lambda(e)^-}$, and define $M_{\bar{e}}$ similarly.
- (4) Try to find an isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ compatible with the intertwiner maps. If it does not exist, return **fail**.
- (5) If e^+ is not an essential sphere vertex, then do the following. If $\lambda(e^+) \notin S$, then (in this case $\lambda(e^+)$ is a curve vertex) choose a biprincipal \mathbb{Z} -biset $M_{\lambda(e^+)}$, choose a legal intertwiner from $M_{\lambda(e)}$ to $M_{\lambda(e^+)}$, and add e^+ to S . Try to find an isomorphism between $M_{\lambda(e^+)} \otimes B_{e^+}$ and $C_{h(e^+)}$ that is compatible with the isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ via the intertwiner maps. If it does not exist, return **fail**.
- (6) If e^+ is an essential sphere vertex, then do the following. Try to find $M_{\lambda(z)} \in \mathbf{Mod}(G_{\lambda(z)})$ such that $M_{\lambda(z)} \otimes B_z \cong C_{h(z)}$. If such $M_{\lambda(z)}$ do not exist, return **fail**. Try to find a legal intertwiner $()^-: M_{\lambda(e)} \rightarrow M_{\lambda(e^+)}$ such that the isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ is compatible

with the isomorphism between $M_{\lambda(e)} \otimes B_e$ and $C_{h(e)}$ via the intertwiner maps. If no such intertwiner exists, return **fail**. Add e^+ to S .

- (7) Return the principal sphere tree of bisets \mathfrak{M} and the isomorphism between \mathfrak{C} and $\mathfrak{M} \otimes \mathfrak{B}$ constructed via h .

Algorithm 4.18. GIVEN ${}_X\mathfrak{B}_X$ and ${}_X\mathfrak{C}_X$ two sphere X -trees of bisets,
 DECIDE whether $\mathfrak{C} \in M(\mathfrak{B})$, and if so CONSTRUCT $\mathfrak{M}, \mathfrak{N} \in \mathbf{Mod}(X)$ such that
 $\mathfrak{C} \cong \mathfrak{M} \otimes \mathfrak{B} \otimes \mathfrak{N}$ AS FOLLOWS:

- (1) Follow Algorithm 6.11 to compute a basis of the mapping class biset $M(\mathfrak{B})$.
- (2) For each \mathfrak{N} in the basis, do the following. Run Algorithm 4.17 on \mathfrak{C} and $\mathfrak{B} \otimes \mathfrak{N}$. If there exists $\mathfrak{M} \in \mathbf{Mod}(X)$ with $\mathfrak{C} \cong \mathfrak{M} \otimes (\mathfrak{B} \otimes \mathfrak{N})$, return $(\mathfrak{M}, \mathfrak{N})$.
- (3) Return **fail**.

5. EXTENSIONS OF BISETS

This section explains how algorithmic problems on bisets, and in particular, conjugacy and computation of centralizers can be carried from subbisets to an extension.

We place ourselves in an abstract setting of group actions. Since a G - H -biset is nothing more than a $(G \times H)$ -set under the rule $(g, h) \cdot b = g \cdot b \cdot h^{-1}$, we consider left G -sets in this section.

Let G be a group, let ${}_G B$ be a left G -set, and let $\phi: H \rightarrow G$ be a group homomorphism. The following two decision problems for $(\phi, {}_G B)$ will be of importance in this section:

The orbit problem: Given $b_1, b_2 \in B$, does there exist $h \in H$ with $h^\phi \cdot b_1 = b_2$? If so, find one.

The stabilizer problem: Given $b \in B$, compute $H_b := \{h \in H \mid h^\phi \cdot b = b\}$.

Typically, G and H will have fixed generating sets, and B will be transitive, hence of the form Gb_0 for some $b_0 \in B$ fixed once and for all. We make this more precise as follows: we are given a class Ω of groups, thought of as “computable” groups. They should be finitely generated, and have solvable word problem, i.e. for each $G \in \Omega$ there is an algorithm that, given $g, h \in G$ as words in a generating set for G , determines whether or not $g = h$. We assume G and H belong to Ω . A subgroup K of a computable group G is *computable* if K is finitely generated and has solvable membership problem (i.e. there is an algorithm that, given $g \in G$, decides whether $g \in K$). A subgroup L of a computable group G is *sub-computable* if there is a computable subgroup $K \leq G$ and a computable homomorphism $K \rightarrow A$ to an Abelian group such that $L = \ker(K \rightarrow A)$. (It follows from the definition that L also has solvable membership problem because it is decidable if $h \in H$ is in $\ker(H \rightarrow A)$.)

We say that the orbit problem is *solvable* if there is an algorithm that, upon receiving $b_1, b_2 \in B$ as input, answers whether there exists an $h \in H$ with $h^\phi b_1 = b_2$, and if so produces one. The elements b_1, b_2 will be given as $g_1 b_0, g_2 b_0$ for words g_1, g_2 in the generating set of G , and h , if it exists, will be returned as a word in the generators of H .

We say that the stabilizer problem is *solvable*, respectively *sub-solvable*, if there is an algorithm that, upon receiving $b \in B$ as input, computes the stabilizer H_b as a computable, respectively sub-computable subgroup of H .

Note that the orbit problem could be reduced to the stabilizer problem as soon as the class Ω is closed under taking products and finite extensions. More precisely, consider an orbit problem $\exists?h \in H : h^\phi \cdot b_1 = b_2$ on a G -set B . Consider the wreath product $G' := G \wr 2\downarrow$ acting naturally on $B \times B$, with $G \times G$ acting coördinatewise on $B \times B$ and $2\downarrow$ acting by permuting the factors of $B \times B$. Define $\phi' : H' := H \wr 2\downarrow \rightarrow G'$ naturally. If the stabilizer $H'_{(b_1, b_2)}$ of (b_1, b_2) is computable in H' , then original orbit problem is solvable: b_1 and b_2 are in the same orbit if and only if some generator of $H'_{(b_1, b_2)}$ permutes both copies of B ; and a witness h to the orbit problem is easily obtained from this generator. We elected not to make use of this reduction, treating the orbit problem as an illustrative step.

5.1. Extensions of G -sets. Let ${}_G B$ be a left G -set and let N be a normal subgroup of G . We have a short exact sequence of groups

$$(13) \quad 1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} G/N \longrightarrow 1.$$

The N -set ${}_N B$ is defined to be B as a set, with the restricted action of N .

Let us denote by $N \backslash ({}_G B)$ the set of connected components of ${}_N B$; namely, its space of N -orbits:

$$N \backslash ({}_G B) := B / \{b_1 \sim b_2 \text{ if and only if } b_1 = nb_2 \text{ for some } n \in N\}.$$

We denote by $\pi : {}_G B \rightarrow N \backslash ({}_G B)$ the induced quotient map. The action of G/N on $N \backslash ({}_G B)$ is given by

$$g^\pi \cdot b^\pi = (gb)^\pi.$$

It is straightforward to verify that $N \backslash ({}_G B)$ is a G/N -set. We will call the sequence

$$(14) \quad {}_N B \hookrightarrow {}_G B \xrightarrow{\pi} N \backslash ({}_G B)$$

an *extension of G -sets*.

5.2. Orbit and stabilizer problems in G -sets. We consider a class Ω of “computable” groups as above. The main step in our program to reduce decision problems on extensions to decision problems on kernel and quotient is contained in the following

Proposition 5.1. *Let ${}_G B$ be a G -set, let N be a central subgroup of G , let*

$$N \xhookrightarrow{\iota} G \xrightarrow{\pi} G/N \quad \text{and} \quad {}_N B \hookrightarrow {}_G B \xrightarrow{\pi} N \backslash ({}_G B)$$

be the corresponding extensions of groups and sets respectively, and let $\phi : H \rightarrow G$ be a group homomorphism. Assume that H, N, G all belong to Ω , and that the orbit and stabilizer problems are solvable in $(\iota, {}_G B)$.

Then there is an algorithm that, given

- *elements $b_1, b_2 \in B$ such that b_1^π and b_2^π are in the same orbit of $N \backslash ({}_G B)$;*
- *an element $h \in H$ such that $b_1^\pi = (h^\phi b_2)^\pi$;*
- *the stabilizer of b_1^π as a computable subgroup of H that belongs to Ω ,*

determines whether b_1, b_2 belong to the same orbit of ${}_G B$, and computes their stabilizers as kernels of maps $\chi : \tilde{H} \rightarrow A$ for a computable $\tilde{H} \leq H$ belonging to Ω and A a finitely generated abelian group.

In particular, if the orbit and stabilizer problems are solvable in $(\phi\pi, N \backslash ({}_G B))$ and in $(\iota, {}_G B)$, then they are sub-solvable in (ϕ, B) .

Proof. We start by the orbit problem in $(\phi, {}_G B)$, and consider $b_1, b_2 \in B$ and $h' \in H$ with $b_1^\pi = ((h')^\phi b_2)^\pi$. Since the orbit problem is solvable in $(\iota, {}_G B)$, an element $a' \in N$ is computable such that $b_1 = a'(h')^\phi b_2$, and it remains to find whether a' may be chosen in H^ϕ .

Consider the stabilizer $H_{b_1^\pi} = \{h \in H \mid (h^\phi b_1)^\pi = b_1^\pi\}$ of b_1^π and the stabilizer $N_{b_1} = \{a \in N \mid ab_1 = b_1\}$ of b_1 ; both are computable. Consider the homomorphism

$$\chi: H_{b_1^\pi} \rightarrow NG_{b_1}/G_{b_1} \cong N/N_{b_1}.$$

It is computable: the quotient N/N_{b_1} is computable as a quotient of finitely generated abelian groups, and for each $h \in H_{b_1^\pi}$ there is a computable $a \in N$ with $h^\phi b_1 = ab_1$, for which $h^\chi = aN_{b_1}$.

We return to the equation $b_1 = a'(h')^\phi b_2$. The orbit problem has a positive solution if and only if there exists $h \in H_{b_1^\pi}$ with $h^\chi = a'N_{b_1}$, in which case $b_1 = (hh')^\phi b_2$, and this is a decidable problem since it takes place in N/N_{b_1} .

Furthermore, the stabilizer of b_1 is the kernel of χ ; so the stabilizer problem for (ϕ, B) is sub-solvable. \square

We remark that the stabilizer cannot be better written than as kernel of map to some abelian group; in particular, it can be infinitely generated, and therefore not directly describable by itself, see §8.3.

5.3. Extensions of left-free bisets. A G_1 - G_2 -biset is a $(G_1 \times G_2)$ -set with a fixed decomposition of $G_1 \times G_2$ as a product of G_1 and G_2 . Namely, if ${}_{G_1}B_{G_2}$ is a biset, then the *associated* $(G_1 \times G_2)$ -set ${}_{G_1 \times G_2}B$ is defined by

$$(15) \quad (g_1, g_2) \cdot b = g_1 b g_2^{-1}.$$

In the inverse direction, if $G_1 \times G_2$ acts on B , then the G_1 - and G_2 -actions on B commute and (15) defines a G_1 - G_2 -biset structure on B .

If N_1, N_2 are normal subgroups of G_1, G_2 respectively, then the N_1 - N_2 -biset ${}_{N_1}B_{N_2}$ and the quotient G_1/N_1 - G_2/N_2 -biset $N_1 \backslash ({}_{G_1}B_{G_2})/N_2$ are defined as for G -sets using (15).

Definition 5.2 (Extensions of bisets). Let ${}_{G_1}B_{G_2}$ be a G_1 - G_2 -biset and let N_1, N_2 be normal subgroups of G_1 and G_2 respectively, so that for $i = 1, 2$ we have short exact sequences

$$(16) \quad 1 \longrightarrow N_i \longrightarrow G_i \xrightarrow{\pi} Q_i \longrightarrow 1.$$

If the quotient Q_1 - Q_2 -biset $N_1 \backslash B/N_2$, consisting of connected components of ${}_{N_1}B_{N_2}$, is left-free, then the sequence

$$(17) \quad {}_{N_1}B_{N_2} \hookrightarrow {}_{G_1}B_{G_2} \xrightarrow{\pi} {}_{Q_1}(N_1 \backslash B/N_2)_{Q_2}$$

is called an *extension of left-free bisets*. \triangle

The following statement follows directly from Definition 5.2.

Proposition 5.3. *Let ${}_{G_1}B_{G_2}$ be a left-free G_1 - G_2 -biset and let N_1, N_2 be normal subgroups of G_1, G_2 respectively. Then the sequence (17) is an extension of left-free bisets if and only if the following property holds:*

$$\text{if } g_1 \in G_1, n_2 \in N_2, b \in B, \text{ and } g_1 b n_2 = b, \text{ then } g_1 \in N_1.$$

Let ${}_G B_G$ be a G -biset. We are interested in the following decision problems in B :

The conjugacy problem: Given $b_1, b_2 \in B$, does there exist $g \in G$ with $g \cdot b_1 = b_2 \cdot g$? If so, find one.

The centralizer problem: Given $b \in B$, compute

$$Z(b) := \{g \in G \mid g \cdot b = b \cdot g\}.$$

To solve these problems, we remark that conjugacy and centralizer problems in bisets are special cases of the orbit and stabilizer problems in G -sets, using (15) with $H = G$ and $\phi: H \rightarrow G \times G$ given by $g^\phi = (g, g)$. We will use the same conventions for conjugacy and stabilizer problems in bisets as for orbit and stabilizer problems in G -sets.

Example 5.4. A natural source of examples of bisets extensions appears as follows. Consider for $i = 1, 2$ path-connected topological bundles

$$(18) \quad \begin{array}{ccc} F_i & \hookrightarrow & E_i \\ & & \downarrow p_i \\ & & X_i \end{array}$$

with path-connected fibres; fix basepoints $*_i \in F_i \subseteq E_i$, so $F_i = p_i^{-1}(p_i(*_i))$. Consider the fundamental groups $N_i = \pi_1(F_i, *_i)$, $G_i = \pi_1(E_i, *_i)$ and $Q_i = \pi_1(X_i, p_i(*_i))$. Then we have short exact sequences of groups $N_i \hookrightarrow G_i \twoheadrightarrow Q_i$ provided that the connecting homomorphisms $\beta_i: \pi_2(X_i) \rightarrow N_i$ are trivial (alternatively we may define N_i as $\pi_1(F_i, *_i)/\beta_i(\pi_2(X_i))$). Let there also be given a bundle partially-defined covering map $(f: E_1 \dashrightarrow E_2, g: X_1 \dashrightarrow X_2)$, with $p_2 \circ f = g \circ p_1$. Recall (as in (4))

$$(19) \quad B(f) = \{\beta: [0, 1] \rightarrow E_1 \mid \beta(0) = *_1, f(\beta(1)) = *_2\} / \approx.$$

We then have a short exact sequence of bisets

$$N_1 B(f)_{N_2} \hookrightarrow G_1 B(f)_{G_2} \twoheadrightarrow Q_1(N_1 \backslash B(f) / N_2)_{Q_2},$$

which are all left-free by Proposition 5.3, since for every loop in a fibre F_2 its f -lifts are all confined to fibres. Furthermore,

Lemma 5.5. *Suppose that $p_1^{-1}(y) \cap f^{-1}(p_2^{-1}(f(y)))$ has a single path connected component. Then the natural map $\beta \mapsto p_1 \circ \beta$ defines a Q_1 - Q_2 -biset isomorphism*

$$N_1 \backslash B(f) / N_2 \xrightarrow{\cong} B(g).$$

Proof. Clearly, $\beta \mapsto p_1 \circ \beta$ defines an epimorphism; we need to show that it is also injective. Consider $\beta, \beta' \in B(f)$ with $p_1 \circ \beta \approx p_1 \circ \beta'$. Then β, β' ends at the same fiber F' of $p_1: E_1 \twoheadrightarrow X_1$. We may choose $n \in N_2$ such that $\beta = \beta' n_2$. Finally there exists an $n_1 \in N_1$ such that $n_1 \beta n_2 = \beta'$ (because $p_1 \circ \beta \approx p_1 \circ \beta'$). \square

6. MAPPING CLASS GROUPS AND MAPPING CLASS BISETS

First, we recall basic properties of mapping class groups (for more details, see [15] and [18]) and the solution of the conjugacy problem in mapping class groups.

6.1. Mapping class groups. Consider a marked sphere (S^2, A) , with A the set of marked points, and a multicurve $\mathcal{C} \subset (S^2, A)$, see §3.1. Let

- $\mathbf{Mod}(S^2, A)$ be the pure mapping class group of (S^2, A) ; it is the set of homeomorphisms $h: S^2 \hookrightarrow$ which fix A , considered up to isotopy rel A ;
- $\mathbf{Mod}(S^2, A, \mathcal{C})$ be the maximal subgroup of $\mathbf{Mod}(S^2, A)$ that fixes each curve of \mathcal{C} up to isotopy;
- $\mathbf{eMod}(S^2, A, \mathcal{C})$ be the subgroup of $\mathbf{Mod}(S^2, A, \mathcal{C})$ generated by Dehn twists around \mathcal{C} ;
- $\mathbf{vMod}(S^2, A, \mathcal{C})$ be the pure mapping class group of $(S^2, A) \setminus \mathcal{C}$.

Their meaning is the following: \mathbf{Mod} is the classical mapping class group, fixing a marked set and possibly a multicurve. We remark that $\mathbf{Mod}(S^2, A)$ could equally well have been defined as *homotopy* classes of homeomorphisms, see [14, §6]. \mathbf{eMod} is the group of mapping classes that act only on edges of the decomposition of S^2 along \mathcal{C} . Similarly, \mathbf{vMod} is the group of mapping classes that act disjointedly on vertices of the decomposition of S^2 along \mathcal{C} .

It is known [15, Lemma 3.11] that $\mathbf{eMod}(S^2, A, \mathcal{C})$ is isomorphic to $\mathbb{Z}^{\#\mathcal{C}}$ and is in the center of $\mathbf{Mod}(S^2, A, \mathcal{C})$. Furthermore, there is a central exact sequence of groups

$$(20) \quad 1 \longrightarrow \mathbf{eMod}(S^2, A, \mathcal{C}) \xrightarrow{\iota} \mathbf{Mod}(S^2, A, \mathcal{C}) \xrightarrow{\pi} \mathbf{vMod}(S^2, A, \mathcal{C}) \longrightarrow 1$$

which is easily seen to be split, see [15, Proposition 3.20].

Theorem 6.1 (Nielsen [28], Thurston [32]). *Every homeomorphism $f: (S^2, A) \hookrightarrow$ may be isotoped to a homeomorphism $g: (S^2, A) \hookrightarrow$ that either*

- *is periodic;*
- *preserves a multicurve on (S^2, A) (in this case, g is called reducible); or*
- *is pseudo-Anosov.*

A pseudo-Anosov homeomorphism is never periodic nor reducible. □

Furthermore, there are algorithms, such as “train track” technology (see [8]) that determine to which class f belongs, and compute the multicurve along which f can be reduced.

Let \mathcal{C}_f be the minimal multicurve that is invariant by f and such that all periodic components of $f: (S^2, A) \setminus \mathcal{C}_f \hookrightarrow$ are, up to isotopy, either pseudo-Anosov or of finite order. The multicurve \mathcal{C}_f is unique, so f belongs to $\mathbf{Mod}(S^2, A, \mathcal{C}_f)$ because a mapping class permuting non-trivially a multicurve must also permute the marked points.

The following algorithm solves the conjugacy problem under $\mathbf{Mod}(S^2, A)$ for homeomorphisms $f, g: (S^2, A) \hookrightarrow$. The problem of combinatorial equivalence between homeomorphisms $(S^2, A) \hookrightarrow$ reduces to the pure case because the non-pure mapping class group of (S^2, A) is a finite extension of $\mathbf{Mod}(S^2, A)$. The extension of this algorithm to non-invertible maps is the main outcome of this series of articles.

Algorithm 6.2. GIVEN *homeomorphisms $f, g: (S^2, A) \hookrightarrow$* ,
COMPUTE *whether they are conjugate by $\mathbf{Mod}(S^2, A)$* , and COMPUTE *the centralizer of f in $\mathbf{Mod}(S^2, A)$* AS FOLLOWS:

- (1) Compute the canonical multicurves $\mathcal{C}_f, \mathcal{C}_g$. They may be constructed e.g. using the Bestvina-Handel “train track” algorithm [8].

- (2) Check whether there is an $h \in \mathbf{Mod}(S^2, A)$ such that h conjugates $g: \mathcal{C}_g \hookrightarrow$ to $f: \mathcal{C}_f \hookrightarrow$. If h does not exist, then f and g are not conjugate by $\mathbf{Mod}(S^2, A)$. Otherwise choose one such h (all such h differ by post-composition by elements in $\mathbf{Mod}(S^2, A, \mathcal{C}_f)$) and replace g by g^h . Then $\mathcal{C}_f = \mathcal{C}_g =: \mathcal{C}$ and the conjugacy problem for f, g under $\mathbf{Mod}(S^2, A)$ and under $\mathbf{Mod}(S^2, A, \mathcal{C})$ are equivalent.

- (3) Check whether the projections $\pi(f), \pi(g) \in \mathbf{vMod}(S^2, A, \mathcal{C})$ are conjugate under $\mathbf{vMod}(S^2, A, \mathcal{C})$. Every periodic component of $\pi(f)$ and $\pi(g)$ is either of finite order or pseudo-Anosov; and in both of these cases the conjugacy and centralizer problems are solvable, see [18].

If the projections are conjugate, choose an element $q \in \mathbf{Mod}(S^2, A, \mathcal{C})$ such that $\pi(f) = \pi(g^q)$. Since $\mathbf{eMod}(S^2, A)$ belongs to the centre of $\mathbf{Mod}(S^2, A, \mathcal{C})$, the rotation parameter $f^{-1}g^q \in \mathbf{eMod}(S^2, A, \mathcal{C})$ does not depend on the choice of q .

- (4) The maps f, g are conjugate under $\mathbf{Mod}(S^2, A, \mathcal{C})$ if and only if $f^{-1}g^q$ is the identity in $\mathbf{eMod}(S^2, A, \mathcal{C})$, and a conjugator is for example q .
- (5) The centralizer $Z(f)$ is a subgroup of $\mathbf{Mod}(S^2, A, \mathcal{C})$, and we describe it in terms of (20). Its intersection with $\mathbf{eMod}(S^2, A, \mathcal{C}) \cong \mathbb{Z}^{\mathcal{C}}$ is easy to compute: it consists of those maps $\mathcal{C} \rightarrow \mathbb{Z}$ that are constant on every f -orbit in \mathcal{C} .

Consider next the image of $Z(f)$ in $\mathbf{vMod}(S^2, A, \mathcal{C})$. Let $\hat{S}_0, \hat{S}_1 := f(\hat{S}_0), \dots, \hat{S}_n = \hat{S}_0$ be a cycle of small spheres under $\pi(f)$; for simplicity, write $f_i: \hat{S}_i \rightarrow \hat{S}_{i+1}$, with indices computed modulo n , for the restrictions of f to the small spheres; they are all easily computable.

Elements of $Z(f)$ must be constant on orbits of small spheres in the sense that, if $z_i: \hat{S}_i \hookrightarrow$ are the coördinates of an element centralizing f , then $z_i^{f_i} = z_{i+1}$, indices again read modulo n . Compute the return map $f' = f_0 f_1 \dots f_{n-1}: \hat{S}_0 \hookrightarrow$. Then, by our assumption, the map f' is either of finite order or pseudo-Anosov, and this can be determined by checking if $(f')^e = 1$ for the smallest e such that $(f')^e$ is pure. Suppose that f' is of finite order. Consider the quotient surface \hat{S}_0/f' ; it is also a sphere, with up to two extra punctures corresponding to the centres of rotation of f' . Then $Z(f')$ is isomorphic to a finite-index subgroup of $\mathbf{Mod}(\hat{S}_0/f')$: every $z_0 \in Z(f')$ projects via $\hat{S}_0 \rightarrow \hat{S}_0/f'$ to an element in $\mathbf{Mod}(\hat{S}_0/f')$, and conversely every $h \in \mathbf{Mod}(\hat{S}_0/f')$ lifts to e possibly non-pure mapping classes commuting with f , namely $\tilde{h}, f\tilde{h}, \dots, f^{e-1}\tilde{h}$, and at most one of these is pure.

Let us now briefly consider the case that f' is pseudo-Anosov; details will appear in [6]. Since f' preserves transverse foliations on \hat{S}_0 , its centralizer coincides with its *radical*, $\sqrt{f'} = \{z \in \mathbf{Mod}(\hat{S}_0) \mid \exists k, \ell \neq 0 \text{ with } z^k = (f')^\ell\}$. The map f' admits a *train track representation*, namely there exists a 1-dimensional submanifold with forks on \hat{S}_0 , such that f' and in fact all elements of $\sqrt{f'}$ may be represented by non-negative integer labellings of the train track subject to some arithmetic conditions. It follows that $\sqrt{f'}$ is a cyclic subgroup, and since the numbers on the train track increase upon taking powers, a generator of $\sqrt{f'}$ may be found by enumerating all labellings with numbers less than those of f' to find a minimal root of f' .

6.2. Mapping class bisets. We now turn to maps of degree > 1 . Kameyama introduces in [20, §4] a semigroup containing $\mathbf{Mod}(S^2, A)$,

$$K(S^2, A) = \{\text{Thurston maps } (S^2, A) \hookrightarrow\} / \approx$$

and derives some important properties. Clearly $K(S^2, A)$ is a $\mathbf{Mod}(S^2, A)$ -biset, under pre- and post-composition of Thurston maps. It is a graded semigroup under the map $\deg: K(S^2, A) \rightarrow \mathbb{N}^*$, and is finitely generated in every degree.

We will need the more general situation afforded by non-dynamical maps, i.e. maps with different domain and range. Let therefore $f: (S^2, C) \rightarrow (S^2, A)$ be a sphere map, and let \mathcal{D}, \mathcal{C} be multicurves on $S^2 \setminus C, S^2 \setminus A$ respectively with $\mathcal{D} \subseteq f^{-1}(\mathcal{C})$.

Definition 6.3 (Mapping class bisets). The $\mathbf{Mod}(S^2, C)$ - $\mathbf{Mod}(S^2, A)$ -biset $M(f, C, A)$ is defined as

$$M(f, C, A) = \{m' f m'' \mid m' \in \mathbf{Mod}(S^2, C), m'' \in \mathbf{Mod}(S^2, A)\} / \approx.$$

It admits as a subbiset the $\mathbf{Mod}(S^2, C, \mathcal{D})$ - $\mathbf{Mod}(S^2, A, \mathcal{C})$ -biset

$$M(f, C, A, \mathcal{D}, \mathcal{C}) = \{m' f m'' \mid m' \in \mathbf{Mod}(S^2, C, \mathcal{D}), m'' \in \mathbf{Mod}(S^2, A, \mathcal{C})\} / \approx.$$

The left and right actions are given by $m' f m'' = m'' \circ f \circ m'$, in keeping with using the algebraic order of operations in bisets. \triangle

In case $G = \pi_1(S^2 \setminus A, *)$ is a sphere group, we write $\mathbf{Mod}(G)$ for $\mathbf{Mod}(S^2, A)$; by Theorem 2.3, it is a subgroup of the outer automorphism group of G , and is defined algebraically in terms of G and its peripheral conjugacy classes. If \mathfrak{X} be a sphere tree of groups, we denote by $\mathbf{Mod}(\mathfrak{X})$ its group of pure self-congruences, so that if \mathcal{C} is a multicurve on (S^2, A) then $\mathbf{Mod}(\mathfrak{X}) \cong \mathbf{Mod}(S^2, A, \mathcal{C})$ for the tree of groups decomposition \mathfrak{X} of G along \mathcal{C} . Similarly, we write

$$K(G) = \{G\text{-}G\text{-bisets}\} / \cong$$

for the $\mathbf{Mod}(G)$ -biset isomorphism classes of bisets of Thurston maps $(S^2, A) \hookrightarrow$, which by Theorem 2.8 is in bijection with $K(S^2, A)$.

If B be a sphere H - G -biset, then by $M(B)$ we denote the $\mathbf{Mod}(H)$ - $\mathbf{Mod}(G)$ -biset

$$M(B) = \{B_\psi \otimes B \otimes B_\phi \mid \psi \in \mathbf{Mod}(H), \phi \in \mathbf{Mod}(G)\} / \cong.$$

By Theorem 2.8, for a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ the bisets $M(f, C, A)$ and $M(B(f))$ are isomorphic, so $M(f, C, A)$ can also be viewed as the biset of isomorphism classes of bisets of the form $B(m') \otimes B(f) \otimes B(m'')$ with $m' \in \mathbf{Mod}(S^2, C), m'' \in \mathbf{Mod}(S^2, A)$,

Similarly, if \mathfrak{B} be an \mathfrak{Y} - \mathfrak{X} -tree of bisets, we define

$$M(\mathfrak{B}) = \{\mathfrak{N}' \otimes \mathfrak{B} \otimes \mathfrak{N}'' \mid \mathfrak{N}' \in \mathbf{Mod}(\mathfrak{Y}), \mathfrak{N}'' \in \mathbf{Mod}(\mathfrak{X})\} / \cong,$$

and note that for a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ the bisets $M(f, C, A, \mathcal{D}, \mathcal{C})$ and $M(\mathfrak{B}(f))$ are isomorphic. This may be seen in a manner that makes more explicit the tree of bisets decomposition of f : let $\mathfrak{Y} \mathfrak{B} \mathfrak{X}$ be the tree of bisets decomposition of f . Every element $b \in M(f, C, A, \mathcal{D}, \mathcal{C})$ is encoded by a well-defined tree of bisets $\mathfrak{Y} \mathfrak{C} \mathfrak{X}$ over the same trees of groups $\mathfrak{Y}, \mathfrak{X}$; so $M(f, C, A, \mathcal{D}, \mathcal{C})$ is the biset of isomorphism classes of such trees of bisets.

6.2.1. Properties of Mapping class bisets.

Proposition 6.4 (An extension of [20, Proposition 4.1]). *The bisets $M(f, C, A)$ and $M(f, C, A, \mathcal{D}, \mathcal{C})$ are left-free.*

Proof. The second claim obviously follows from the first. Consider $g \in M(f, C, A)$ and $m \in \mathbf{Mod}(S^2, C)$ with $mg = g$. We use the same letters g, m to denote a representative Thurston map and homeomorphism respectively; so g and $g \circ m$ are isotopic along an isotopy fixing C . Lifting the isotopy from g to $g \circ m$ gives $g = g \circ m'$ for some homeomorphism m' isotopic to m . We wish to show that m' is isotopic to the identity.

Consider a path $\gamma: [0, 1] \rightarrow S^2 \setminus A$ with $\gamma(0) \in A$, possibly a critical value; and consider its g -preimages $\gamma_1, \dots, \gamma_d$. They are permuted by m' , in the sense that $m' \circ \gamma_i \in \{\gamma_1, \dots, \gamma_d\}$, so there exists $e \in \{1, \dots, d\}$ such that $(m')^e \circ \gamma_1 = \gamma_1$ and in particular $(m')^e(\gamma_1(1)) = \gamma_1(1)$. This continues to hold as γ_1 is deformed, and $\gamma_1(1)$ is arbitrary, so $(m')^e = 1$, and m' being of finite order is isotopic to a Möbius transformation by Theorem 6.1. If $\#C \geq 3$, we are done since a Möbius transformation fixing three points is trivial, while if $\#C = 2$ then m' is isotopic to a rotation on a sphere with two marked points, and again is isotopic to the identity. \square

Example 6.5. Let us note that the non-pure version of the mapping class biset needs not be left-free; here is the simplest example. Consider the sphere map $f(z) = z^2: (\widehat{\mathbb{C}}, \{-1, 1\}) \rightarrow (\widehat{\mathbb{C}}, \{0, 1, \infty\})$. The non-pure mapping class group $\mathbf{Mod}^*(\widehat{\mathbb{C}}, \{-1, 1\})$ of $(\widehat{\mathbb{C}}, \{-1, 1\})$ consists of two elements $1, z \rightarrow -z$. Since $z^2 \circ [z \rightarrow -z] = z^2$ the left action of $\mathbf{Mod}^*(\widehat{\mathbb{C}}, \{-1, 1\})$ on $\{f\}$ is trivial.

We extract, out of a left-free biset, relevant information from which a basis of mapping class bisets can be constructed.

Definition 6.6 (Distillations). Let G be a sphere group with fixed Hurwitz generators $\gamma_1, \dots, \gamma_n$. Let ${}_H B_G$ be a sphere biset, and choose a basis S of B . Consider the *wreath map* associated with S , see [3, §2.5]: the map $\Phi: G \rightarrow H \wr S$ such that $s \cdot g = h \cdot t$ if $\Phi(g) = \langle \dots, h, \dots \rangle \pi$ with $s^\pi = t$ and the ‘ h ’ in position s .

Write $\Phi(\gamma_i) = \langle h_{i,1}, \dots, h_{i,d} \rangle \pi_i$. For each cycle $C_{i,j}$ of π_i , let $\widehat{h_{i,j}}$ be the conjugacy class, in H , of the product of the $h_{i,k}$ along the cycle. Then each $\widehat{h_{i,j}}$ is either trivial or a peripheral conjugacy class in H . We call the collection $\overline{\Phi} = \{\pi_i, \widehat{h_{i,j}}\}$ the *distillation* of Φ .

Note that we consider distillations up to the diagonal action by conjugation by a permutation of S , since the set S is not assumed ordered. Note also that there are finitely many distillations, since there are finitely many choices for the permutations π_i and the conjugacy classes $\widehat{h_{i,j}}$. Furthermore, all peripheral conjugacy classes appear exactly once among the $\widehat{h_{i,j}}$, by Definition 2.6(SB₃). \triangle

Lemma 6.7. *Let $M(B_0)$ be the mapping class biset of a sphere biset ${}_H(B_0)_G$, and consider a biset $B \in M(B_0)$, with wreath recursion Φ . Then the distillation $\overline{\Phi}$ depends only on B , and is called the distillation \overline{B} of B .*

Let $B, C \in M(B_0)$ be bisets. Then $\overline{B} = \overline{C}$ if and only if B, C belong to the same left orbit of $M(B_0)$.

Proof. For the first statement, the multiset of lifts, and the monodromy actions of the Hurwitz generators, are invariants of B .

For the second claim, let Φ, Ψ be wreath recursions of B, C respectively. If $\overline{\Phi} = \overline{\Psi}$ then they have the same permutations π_i , up to relabelling, and therefore the same cycles $C_{i,j}$. Let $h'_{i,j}$ be the product of the entries of $\Phi(\gamma_i)$ along the cycle $C_{i,j}$ and let $h''_{i,j}$ be the product of the entries of $\Psi(\gamma_i)$ along the same cycle $C_{i,j}$, in the same order. Then $h'_{i,j}$ and $h''_{i,j}$ are conjugate for each i, j , and the map $h'_{i,j} \mapsto h''_{i,j}$ extends to an automorphism of H , because the only relation in H is that the product of the $h'_{i,j}$, in some order, is trivial, and the product of the $h''_{i,j}$, in the same order, is also trivial. This automorphism is defined up to inner automorphisms. Since it preserves the peripheral conjugacy classes, it defines an element $m' \in \mathbf{Mod}(H)$, again by Theorem 2.3, and we have $\Psi = \Phi(m' \times \cdots \times m')$, so $B = mC$.

Conversely, if $B = mC$ for $m \in \mathbf{Mod}(H)$ then obviously $\overline{B} = \overline{C}$. \square

Proposition 6.8 (Essentially [22, Proposition 3.1]). *The biset $M(f, C, A)$ has finitely many left $\mathbf{Mod}(S^2, C)$ -orbits, and the biset $M(f, C, A, \mathcal{D}, \mathcal{C})$ has finitely many left $\mathbf{Mod}(S^2, C, \mathcal{D})$ -orbits.*

Proof. For the first claim, choose a basis S of $B(f)$, write $G = \pi_1(S^2 \setminus A, *)$ and $H = \pi_1(S^2 \setminus C, \dagger)$, and $B_0 = B(f)$. The claim then follows from Lemma 6.7.

The second statement follows from the first: since the right action of $\mathbf{Mod}(S^2, A)$ is transitive on $1 \otimes M(f, C, A)$, this last set is of the form $N \backslash \mathbf{Mod}(S^2, A)$ for a finite-index subgroup $N \leq \mathbf{Mod}(S^2, A)$. Then $1 \otimes M(f, C, A, \mathcal{D}, \mathcal{C})$ is isomorphic, qua right $\mathbf{Mod}(S^2, A, \mathcal{C})$ -set, to $(N \cap \mathbf{Mod}(S^2, A, \mathcal{C})) \backslash \mathbf{Mod}(S^2, A, \mathcal{C})$, and $[\mathbf{Mod}(S^2, A, \mathcal{C}) : N \cap \mathbf{Mod}(S^2, A, \mathcal{C})] \leq [\mathbf{Mod}(S^2, A) : N] < \infty$. \square

The biset $M(f, C, A, \mathcal{D}, \mathcal{C})$ restricts to an $\mathbf{eMod}(S^2, C, \mathcal{D})$ - $\mathbf{eMod}(S^2, A, \mathcal{C})$ -biset, which furthermore encodes the Thurston matrix of f , namely the map $T_{f, \mathcal{C}}$ from §3.2.

Proposition 6.9. *Assume $\mathcal{D} = f^{-1}(\mathcal{C})$ and consider $g \in M(f, C, A, \mathcal{D}, \mathcal{C})$. If $e \in \mathbf{eMod}(S^2, A, \mathcal{C})$ and $m \in \mathbf{Mod}(S^2, C, \mathcal{D})$ satisfy $ge = mg$, then $m \in \mathbf{eMod}(S^2, C, \mathcal{D})$.*

Furthermore, identify $\mathbf{eMod}(S^2, A, \mathcal{C})$ with $\mathbb{Z}^{\mathcal{C}}$ and $\mathbf{eMod}(S^2, A, \mathcal{D})$ with $\mathbb{Z}^{\mathcal{D}}$ by writing these groups as generated by Dehn twists. If $ge = mg$, then we have $m = T_{f, \mathcal{C}}(e)$.

Proof. Suppose $m \notin \mathbf{eMod}(S^2, C, \mathcal{D})$. Then there is a small sphere U in $S^2 \setminus \mathcal{D}$ such that the restriction of m to \hat{U} , call it $\hat{m} \in \mathbf{Mod}(\hat{U})$, is non-trivial. As in Definition 4.9, denote by $\hat{g}: \hat{U} \rightarrow \hat{S}$ the small sphere map induced by g . By Proposition 6.4 we have $\hat{g} \neq \hat{m}\hat{g}$, and this contradicts $\hat{g} = \hat{g}e$.

Denote by d the degree of f . Clearly, $m = T_{f, \mathcal{C}}(e)$ if and only if $m^{d!} = T_{f, \mathcal{C}}(e^{d!})$, since $T_{f, \mathcal{C}}$ is a linear operator on free abelian groups.

Consider a curve $\gamma \in \mathcal{C}$ and its (possibly isotopic) preimages $\delta_1, \dots, \delta_m$ mapping to γ with degrees d_1, \dots, d_m respectively. Then the $d!$ -th power of a Dehn twist about γ lifts to the product, for $i = 1, \dots, m$, of $d!/d_i$ -th powers of Dehn twists about δ_i . For each peripheral or trivial curve δ_i , the corresponding Dehn twist is trivial in $\mathbf{eMod}(S^2, C, \mathcal{D})$, while powers add for Dehn twists about isotopic curves. We recover precisely (8). \square

Consider two sphere maps $f: (S^2, D) \rightarrow (S^2, C)$ and $g: (S^2, C) \rightarrow (S^2, A)$. Then there is a natural epimorphism $M(f, D, C) \otimes M(g, C, A) \twoheadrightarrow M(fg, D, A)$ given by composition; however this morphism needs not be injective, see §8.2.1.

Recall [3, §2] that to a group homomorphism $\phi: H \rightarrow G$ we associate the H - G -set B_ϕ , which, qua right G -set, is plainly G ; the left H -action is by

$$h \cdot b = h^\phi b.$$

Suppose that $f: (S^2, C) \rightarrow (S^2, A)$ is a homeomorphism. Then it induces an isomorphism $f_*: \mathbf{Mod}(S^2, C) \rightarrow \mathbf{Mod}(S^2, A)$ and the biset $M(f, C, A)$ is biprincipal and is isomorphic to B_{f_*} . A bit more general:

Lemma 6.10. *Consider a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ of degree 1. Then f induces the forgetful morphism $f^*: \mathbf{Mod}(S^2, A) \rightarrow \mathbf{Mod}(S^2, C)$ so that $fh = f^*(h)f$. The biset $M(f, C, A)$ is isomorphic to B_{f^*} ; it is a left principal and right transitive biset.* \square

6.3. Computability of mapping class bisets. Propositions 6.4 and 6.8 point to the fact that the mapping class biset $M(f, C, A, \mathcal{D}, \mathcal{C})$ is computable. Let us make this more precise.

Firstly, the group $G = \pi_1(S^2 \setminus A, *)$ is computable, since it is a finitely generated free group. The conjugacy problem is easy to solve in G — conjugacy classes are just freely reduced words up to cyclic permutations — so multicurves \mathcal{C} are computable as collections of conjugacy classes.

Secondly, the biset $B(f)$ of a sphere map $f: (S^2, C) \rightarrow (S^2, A)$ is computable, since it is a left-free biset of degree $d = \deg(f)$. All bisets of the form $B(f)$ will be manipulated algorithmically by choosing a basis of cardinality d and working with the wreath map $\Phi: G \rightarrow \pi_1(S^2 \setminus C, \dagger) \wr d!$. The decomposition of $B(f)$ along a multicurve \mathcal{C} is computable by Theorem 3.9.

Thirdly, the group $\mathbf{Mod}(S^2, A)$ is computable, since by Theorem 2.3 it is a subgroup of the outer automorphism group of G . Elements of $\mathbf{Mod}(S^2, A)$ will be manipulated algorithmically as maps $G \hookrightarrow$, by keeping track of their values on the standard generators of G . The condition that a map on generators $\phi: \gamma_i \mapsto w_i$, for $w_i \in G$, defines an element of $\mathbf{Mod}(S^2, A)$ is easy to check: since $\gamma_1 \cdots \gamma_n = 1$ we must have $w_1 \cdots w_n = 1$, and each w_i must be conjugate to γ_i . It is equally easy to check whether ϕ defines an element of a subgroup such as $\mathbf{Mod}(S^2, A, \mathcal{C})$: the corresponding automorphism of G must preserve the conjugacy classes in \mathcal{C} .

Finally, the biset $M(f, C, A)$ and its subbiset $M(f, C, A, \mathcal{D}, \mathcal{C})$ are computable. We express this, following Proposition 6.8, as an

Algorithm 6.11. GIVEN f a sphere map $(S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$,
COMPUTE the biset $M(f, C, A, \mathcal{D}, \mathcal{C})$ AS FOLLOWS:

- (1) Write $G = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n \rangle$ the fundamental group of $S^2 \setminus A$, and identify elements of \mathcal{C} with conjugacy classes in G ; similarly write $H = \langle \delta_1, \dots, \delta_m \mid \delta_1 \cdots \delta_m \rangle$.
- (2) Choose a basis of $B(f)$, and compute its wreath map Φ as a table with n rows and $d+1$ columns, with $d = \deg(f)$, the 0th one being a permutation of $\{1, \dots, d\}$ and the remaining being the entries of $\Phi(\gamma_i)$, written as words in the δ_j .
- (3) Make a list of all *distillations*, see Definition 6.6: data structures consisting of a list of n permutations π_1, \dots, π_n whose product is 1, up to the diagonal action by conjugation by a permutation of $\{1, \dots, d\}$, and for each cycle of each π_i of a conjugacy class $\in \{1, \delta_1^H, \dots, \delta_m^H\}$; for a wreath map Ψ , denote

- by $\overline{\Psi}$ its distillation, obtained by computing the conjugacy class of product of the entries along each cycle.
- (4) View the mapping class group $\mathbf{Mod}(S^2, A)$ as a group of automorphisms of G , considered up to inner automorphisms. It is generated by Dehn twists $\tau_{i,j}$ for all $1 \leq i < j \leq n$; the map $\tau_{i,j}$ conjugates γ_k by $\gamma_i \cdots \gamma_j$ for all $k \in \{i, \dots, j\}$ and fixes the other generators.
 - (5) Start with $X = \{\Phi\}$, the wreath map of f , and $\overline{X} = \{\overline{\Phi}\}$. As long as there exists a generator $\tau_{i,j} \in \mathbf{Mod}(S^2, A)$ and a wreath map $\Psi \in X$ such that $\overline{\tau_{i,j}\Psi} \notin \overline{X}$, add $\tau_{i,j}\Psi$ to X and add $\overline{\tau_{i,j}\Psi} \notin \overline{X}$ to \overline{X} .
 - (6) The set X is now a basis for $M(f, C, A)$, and the wreath map of $M(f, C, A)$ may be computed as follows. For each $m \in \mathbf{Mod}(S^2, A)$ and each $\Psi \in X$, let $\Psi' \in X$ be such that $\overline{\Psi'} = \overline{m\Psi}$. Write $h'_{i,j}$, respectively $h''_{i,j}$, for the products of entries along the cycles of $m\Psi, \Psi'$ respectively. Let $m': H \hookrightarrow H$ be the automorphism of H defined by $h'_{i,j} \mapsto h''_{i,j}$; it may e.g. be normalized by computing the images of the standard generators δ_k . Then the biset $M(f, C, A)$ is defined by the equations $\Psi \cdot m = m' \cdot \Psi'$.
 - (7) The biset $M(f, C, A, \mathcal{D}, \mathcal{C})$ is a subbiset of $M(f, C, A)$ in the following way: rather than saturating X under the action of all $\tau_{i,j}$, only consider those Dehn twists along simple closed curves that do not intersect \mathcal{C} .

6.4. Extensions of mapping class bisets. Since we are mainly interested in the dynamical situation of a Thurston map, let us abbreviate

$$M(f, A) := M(f, A, A), \quad M(f, A, \mathcal{C}) := M(f, A, A, \mathcal{C}, \mathcal{C}).$$

Let us denote by $eM(f, A, \mathcal{C})$ the biset $M(f, A, \mathcal{C})$ with restricted actions, namely the $\mathbf{eMod}(S^2, A, \mathcal{C})$ -biset

$$eM(f, A, \mathcal{C}) := \mathbf{eMod}(S^2, A, \mathcal{C})M(f, A, \mathcal{C})\mathbf{eMod}(S^2, A, \mathcal{C})$$

and let us define the *small mapping class biset* $vM(f, A, \mathcal{C})$ as the $\mathbf{vMod}(S^2, A, \mathcal{C})$ -biset of restrictions to $S^2 \setminus \mathcal{C}$ as in Definition 4.9:

$$(21) \quad vM(f, A, \mathcal{C}) := \{\text{restriction of } g \text{ to } (S^2, A) \setminus \mathcal{C} : g \in M(f, A, \mathcal{C})\}.$$

It follows from Propositions 5.3, 6.4 and 6.9 that

$$(22) \quad eM(f, A, \mathcal{C}) \hookrightarrow M(f, A, \mathcal{C}) \xrightarrow{\pi} \mathbf{eMod}(S^2, A, \mathcal{C}) \setminus M(f, A, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C})$$

is an extension of left-free bisets. We describe it more concretely as follows. First, we have the natural morphism

$$(23) \quad \mathbf{eMod}(S^2, A, \mathcal{C}) \setminus M(f, A, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C}) \rightarrow vM(f, A, \mathcal{C})$$

which maps every $g \in M(f, A, \mathcal{C})$ to its restriction to $S^2 \setminus \mathcal{C}$. This morphism is finite-to-one, because both bisets are left-free and the source biset has finitely many left orbits.

Lemma 6.12. *There is an algorithm with oracle that, OUT OF a solution to the conjugacy and centralizer problems in the range of (23), SOLVES the conjugacy and centralizer problems in the source of (23).*

Proof. Let us denote by π the map (23). Consider $g_1, g_2 \in \mathbf{eMod}(S^2, A, \mathcal{C}) \setminus M(f, A, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C})$ and their images $g_1^\pi, g_2^\pi \in vM(f, A, \mathcal{C})$. If g_1^π and g_2^π are not conjugate, then neither are g_1 and g_2 .

Otherwise, let $h \in \mathbf{vMod}(A, \mathcal{C})$ be such that $hg_1^\pi = g_2^\pi h$. Then g_2^h belongs to the finite set F of π -preimages of g_1^π . Furthermore, the centralizer of g_1^π is computable and acts computably on F . The elements g_1 and g_2 are conjugate if and only if they are in the same orbit under the action of $Z(g_1^\pi)$, and the centralizer of g_1 is the stabilizer of g_1 in F under $Z(g_1^\pi)$, so it has finite index in $Z(g_1^\pi)$ and is computable. \square

Recall that $M(f, A, \mathcal{C})$ is naturally isomorphic to $M(\mathfrak{B})$ if \mathfrak{B} is the sphere tree of bisets of f , see Definition 3.7. Similarly, $eM(f, A, \mathcal{C})$ and $vM(f, A, \mathcal{C})$ can be identified with $eM(\mathfrak{B})$ and $vM(\mathfrak{B})$; for instance, (21) takes form

$$(24) \quad vM(\mathfrak{B}) := \{\text{essential bisets in } \mathfrak{B}\}.$$

6.5. Decomposition of $M(f, C, A, \mathcal{D}, \mathcal{C})$. The operation implicit in (23) may be made more explicit, both at the topological and algebraic levels; we return to the general (non-dynamical) setting of $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$ with $\mathcal{D} = f^{-1}(\mathcal{C}) \text{ rel } C$. The mapping class group \mathbf{Mod} decomposes into a direct product of \mathbf{eMod} and \mathbf{vMod} , see (20). Mapping class bisets encode non-invertible maps, and decompose similarly, as we shall see below. Similarly to (21), (22), and (23) write

$$(25) \quad \begin{aligned} vM(f, C, A, \mathcal{D}, \mathcal{C}) &:= \{\text{restriction of } g \text{ to } (S^2, A) \setminus \mathcal{D} : g \in M(f, C, A, \mathcal{D}, \mathcal{C})\}, \\ \pi: M(f, C, A, \mathcal{C}, \mathcal{D}) &\twoheadrightarrow \mathbf{eMod}(S^2, C, \mathcal{D}) \setminus M(f, C, A, \mathcal{D}, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C}), \\ \sigma: \mathbf{eMod}(S^2, C, \mathcal{D}) \setminus M(f, C, A, \mathcal{D}, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C}) &\rightarrow vM(f, C, A, \mathcal{D}, \mathcal{C}). \end{aligned}$$

Lemma 6.13. *With σ as in (25), there is a finite set S endowed with a right action of $\mathbf{vMod}(S^2, A, \mathcal{C})$ and there is a biset intertwiner*

$$\tau: \mathbf{eMod}(S^2, C, \mathcal{D}) \setminus M(f, C, A, \mathcal{D}, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C}) \rightarrow {}_1 S_{\mathbf{vMod}(S^2, A, \mathcal{C})}$$

such that τ restricts to a bijection $\sigma^{-1}(b) \rightarrow S$ for every $b \in vM(f, C, A, \mathcal{D}, \mathcal{C})$. Endow S with the trivial left action of $\mathbf{vMod}(S^2, C, \mathcal{D})$. Then $\sigma \times \tau$ is a $\mathbf{vMod}(S^2, C, \mathcal{D})$ - $\mathbf{vMod}(S^2, A, \mathcal{C})$ -biset isomorphism

$$\mathbf{eMod}(S^2, C, \mathcal{D}) \setminus M(f, C, A, \mathcal{D}, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C}) \cong vM(f, C, A, \mathcal{D}, \mathcal{C}) \times S.$$

Proof. The set S is the set of restrictions of maps in $\mathbf{eMod}(S^2, C, \mathcal{D}) \setminus M(f, A, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C})$ to all non-essential spheres of $(S^2, f^{-1}(A), f^{-1}(\mathcal{C}))$. More precisely, let $\mathfrak{B}(g)_{\mathfrak{x}}$ be the sphere tree of bisets of $g \in M(f, C, A, \mathcal{D}, \mathcal{C})$, let $\tau(\mathfrak{B}(g))$ be \mathfrak{B} minus all its essential sphere vertex bisets, and let $\sigma(\mathfrak{B}(g))$ be the set of essential sphere bisets in $\mathfrak{B}(g)$.

By construction, $\mathbf{eMod}(S^2, C, \mathcal{D})g = \mathbf{eMod}(S^2, C, \mathcal{D})g'$ if and only if $[\tau(\mathfrak{B}(g)), \sigma(\mathfrak{B}(g))] = [\tau(\mathfrak{B}(g')), \sigma(\mathfrak{B}(g'))]$. We define S as the quotient of $\{\tau(\mathfrak{B}(g)) \mid g \in M(f, C, A, \mathcal{D}, \mathcal{C})\}$ by the right action of $\mathbf{eMod}(S^2, A, \mathcal{C})$. \square

Let

$$(26) \quad H_f := \{e \in \mathbf{eMod}(S^2, A, \mathcal{C}) \mid fe \in \mathbf{eMod}(S^2, C, \mathcal{D})f\}$$

be the subgroup of liftable twists. Denote by Λ the biset of the virtual homomorphism $T_{f, \mathcal{C}}: \mathbf{eMod}(S^2, A, \mathcal{C}) \cong \mathbb{Z}^{\mathcal{C}} \dashrightarrow \mathbb{Z}^{\mathcal{D}} \cong \mathbf{eMod}(S^2, C, \mathcal{D})$ with $\text{Dom}(T_{f, \mathcal{C}}) = H_f$; the $\mathbb{Z}^{\mathcal{C}}$ - $\mathbb{Z}^{\mathcal{D}}$ -biset Λ may be viewed as the abelian group

$$(27) \quad \Lambda := \mathbb{Z}^{\mathcal{D}} \times \mathbb{Z}^{\mathcal{C}} / \{(T_{f, \mathcal{C}}(m), -m) \mid m \in H_f\}$$

with natural actions. Since the action of $\mathbf{eMod}(S^2, C, \mathcal{D}) \times \mathbf{eMod}(S^2, A, \mathcal{C})$ on $M(f, C, A, \mathcal{D}, \mathcal{C})$ commutes with the actions of $\mathbf{vMod}(S^2, C, \mathcal{D})$ and of $\mathbf{vMod}(S^2, A, \mathcal{C})$, the group Λ has a well defined left action on $M(f, C, A, \mathcal{D}, \mathcal{C})$ defined by $\lambda \cdot b = mbn$ if $\lambda = [(m, n)]$ with $m \in \mathbb{Z}^{\mathcal{D}} \cong \mathbf{eMod}(S^2, C, \mathcal{D})$ and $n \in \mathbb{Z}^{\mathcal{C}} \cong \mathbf{eMod}(S^2, A, \mathcal{C})$. We write the action of Λ on $M(f, C, A, \mathcal{D}, \mathcal{C})$ as a left action because of the

Lemma 6.14. *The action of $\Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D})$ on $M(f, C, A, \mathcal{D}, \mathcal{C})$ is free.*

Proof. The group Λ is defined in such a manner that its action on $M(f, C, A, \mathcal{D}, \mathcal{C})$ is free. It follows from Proposition 6.9 that the combined action of $\Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D})$ on $M(f, C, A, \mathcal{D}, \mathcal{C})$ is free. \square

Remark 6.15. It may happen that $T_{f, \mathcal{C}}(e) \in \mathbb{Z}^{\mathcal{D}}$ for some $e \in \mathbb{Z}^{\mathcal{C}}$ that does not correspond to a liftable element, i.e. $e \notin H_f$. In that case, there is no reason for the equality $T_{f, \mathcal{C}}(e)f = fe$ to hold. However, some positive power of e is liftable: if f have degree d , then $T_{f, \mathcal{C}}(e^{d!})f = fe^{d!}$.

It follows that the group Λ need not be free Abelian: the element $(T_{f, \mathcal{C}}(e), -e)$ has finite order in Λ if $e \in \mathbf{eMod}(S^2, A, \mathcal{C}) \setminus H_f$ and $T_{f, \mathcal{C}}(e) \in \mathbb{Z}^{\mathcal{D}}$.

Examples of maps for which this phenomenon occurs include maps f for which $T_{f, \mathcal{C}}$ is integral but some curves in \mathcal{C} have lifts of degree > 1 .

Let

$$(28) \quad G_f := \{m \in \mathbf{vMod}(S^2, A, \mathcal{C}) \mid fm \in (\Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D}))f\}$$

be the group of liftable elements. Then for every $m \in G_f$ there is a unique $(\theta_f(m), \tilde{m}) \in \Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D})$ such that $fm = (\theta_f(m), \tilde{m})f$ holds in $M(f, C, A, \mathcal{D}, \mathcal{C})$. The group Λ as well as θ_f are computable using Algorithm 6.11; this is essentially done in Proposition 6.17. We can also easily compute θ_f on the set of Dehn twists in G_f by lifting the corresponding curves, see Example in §8.3. It is easy to see that G_f is a subgroup of

$$\{m \in \mathbf{vMod}(S^2, A, \mathcal{C}) \mid \sigma \circ \pi(f)m \in \mathbf{vMod}(S^2, C, \mathcal{D})\sigma \circ \pi(f)\}.$$

Theorem 6.16. *Consider a sphere map $f: (S^2, C, \mathcal{D}) \rightarrow (S^2, A, \mathcal{C})$, and the corresponding mapping class bisets $M(f, C, A, \mathcal{D}, \mathcal{C})$. Let $vM_0(f) := \mathbf{vMod}(S^2, C, \mathcal{D})\sigma \circ \pi(f)G_f$ be the $\mathbf{vMod}(S^2, C, \mathcal{D})$ - G_f -subbiset of $vM(f, C, A, \mathcal{D}, \mathcal{C})$ containing f ; let $G_f \leq \mathbf{vMod}(S^2, A, \mathcal{C})$ be the groups of liftable elements as in (28); and let Λ be the $\mathbf{eMod}(S^2, C, \mathcal{D})$ - $\mathbf{eMod}(S^2, A, \mathcal{C}) \times G_f$ -Abelian biset of $T_{f, \mathcal{C}}: \mathbf{eMod}(S^2, A, \mathcal{C}) \cong \mathbb{Z}^{\mathcal{C}} \dashrightarrow \mathbb{Z}^{\mathcal{D}} \cong \mathbf{eMod}(S^2, C, \mathcal{D})$ as above with the action of G_f given via θ_f . Then $M(f, C, A, \mathcal{D}, \mathcal{C})$ decomposes as*

$$\mathbf{Mod}(S^2, C, \mathcal{D})(\Lambda \times vM_0(f))_{\mathbf{eMod}(S^2, A, \mathcal{C}) \times G_f} \otimes \mathbf{Mod}(S^2, A, \mathcal{C})$$

by the map sending $(m_1, g_1)f(m_2, g_2)$ to $([(m_1, m_2)], \sigma(\pi(g_1f))) \otimes g_2$. The actions on $(\Lambda \times vM_0(f))$ are given by

$$(m_1, g_1) \cdot (\lambda, g) \cdot (m_2, g_2) = ((m_1 \lambda m_2 \theta_f(g_2), g_1 g g_2).$$

Proof. View first $M(f, C, A, \mathcal{D}, \mathcal{C})$ as a $(\Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D}))$ - $\mathbf{vMod}(S^2, A, \mathcal{C})$ -biset. Since $M(f, C, A, \mathcal{D}, \mathcal{C})$ is left-free, we have

$$M(f, C, A, \mathcal{D}, \mathcal{C}) \cong (\Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D}))fG_f \otimes_{G_f} \mathbf{vMod}(S^2, A, \mathcal{C}).$$

We then have a decomposition of $\Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D})fG_f$ as

$$\mathbf{eMod}(S^2, C, \mathcal{D}) \times \mathbf{vMod}(S^2, C, \mathcal{D})(\Lambda \times \mathbf{vMod}(S^2, C, \mathcal{D})\pi(f)G_f)_{\mathbf{eMod}(S^2, A, \mathcal{C}) \times G_f}$$

with $\mathbf{vMod}(S^2, C, \mathcal{D})\pi(f)G_f$ viewed as a subbiset of $\mathbf{eMod}(S^2, C, \mathcal{D}) \setminus M(f, C, A, \mathcal{D}, \mathcal{C}) / \mathbf{eMod}(S^2, A, \mathcal{C})$. The biset $\mathbf{vMod}(S^2, C, \mathcal{D})\pi(f)G_f$ is isomorphic to $vM_0(f)$. \square

6.6. Conjugacy and centralizers in mapping class bisets. We are ready to apply Proposition 5.1 to solving centralizer and conjugacy problems in $M(f, A, \mathcal{C})$. We consider as class Ω of “computable” groups all finite direct products $G_1 \times \cdots \times G_n$ of groups G_i which are all finite-index subgroups of modular groups $\mathbf{Mod}(S^2, A_i, \mathcal{C}_i)$.

Note that Ω contains pure mapping class groups, cyclic groups (as mapping class groups of a sphere with four marked points separated by a curve) and $\mathbf{SL}_2(\mathbb{Z})$ (as the mapping class group of a sphere with four marked points). Note in particular that all groups in Ω are finitely presented and have solvable word and conjugacy problem.

Proposition 6.17. *Consider the extension of bisets (22) and the natural morphism (23). Then there is an algorithm, with oracle solving the conjugacy and centralizer problems in $vM(f, A, \mathcal{C})$, by which the conjugacy problem is solvable in $M(f, A, \mathcal{C})$ and the centralizer problem is sub-solvable in $M(f, A, \mathcal{C})$.*

Proof. To apply Proposition 5.1 we need to verify the following two conditions:

- given $b \in eM(f, A, \mathcal{C})$, the stabilizer $P_b = \{(a_1, a_2) \in \mathbf{eMod}(S^2, A, \mathcal{C}) \times \mathbf{eMod}(S^2, A, \mathcal{C}) \mid a_1 b = b a_2\}$ is finitely generated and computable,
- given $b_1, b_2 \in eM(f, A, \mathcal{C})$, it is decidable whether there exist $a_1, a_2 \in \mathbf{eMod}(S^2, A, \mathcal{C})$ with $a_1 b_1 = b_2 a_2$.

For $b \in eM(f, A, \mathcal{C})$, consider the subgroup

$$Q_b := \{a_2 \in \mathbf{eMod}(S^2, A, \mathcal{C}) \mid b a_2 \in \mathbf{Mod}(S^2, A, \mathcal{C})b\}.$$

It has finite index in $\mathbf{eMod}(S^2, A, \mathcal{C})$ so is computable, since $M(f, A, \mathcal{C})$ has finitely many left orbits by Proposition 6.8. Furthermore, it follows from $f^{-1}(\mathcal{C}) \subset \mathcal{C}$ that Q_b has a well-defined image in $\mathbf{eMod}(S^2, A, \mathcal{C})$ under

$$\phi_b: a_2 \mapsto \text{the element } a_1 \in \mathbf{eMod}(S^2, A, \mathcal{C}) \text{ such that } a_1 b = b a_2.$$

Therefore, the stabilizer of b has the following description:

$$P_b = \{(\phi_b(a_2)^{-1}, a_2) \in \mathbf{eMod}(S^2, A, \mathcal{C}) \times Q_b\};$$

this shows that P_b is computable.

Consider $b_1, b_2 \in eM(f, A, \mathcal{C})$. It is decidable whether there exists $a_2 \in \mathbf{eMod}(S^2, A, \mathcal{C})$ with $b_2 a_2 = g_1 b_1$ for some $g_1 \in \mathbf{Mod}(S^2, A, \mathcal{C})$. Then b_1 and b_2 are in the same connected component of $eM(f, A, \mathcal{C})$ if and only if $g_1 \in \mathbf{eMod}(S^2, A, \mathcal{C})$; this last question is decidable. \square

6.7. The vertex mapping class biset of a Thurston map. We study further the mapping class biset $vM(f, A, \mathcal{C})$, decomposing it as a product.

Definition 6.18 (Product bisets). Let $(G_i)_{i \in I}$ be a family of groups, let $G = \prod_i G_i$ be their product, let $f: I \hookrightarrow$ be a map, and let $(B_i)_{i \in I}$ be left-free G_i - $G_{f(i)}$ -bisets. The *product biset* of the bisets B_i is the left-free G - G -biset

$$\prod_i B_i = \{(b_i)_{i \in I} \mid b_i \in B_i\}$$

with actions $h \cdot (b_i)_{i \in I} \cdot g = (h_i b_i g_{f(i)})_{i \in I}$. \triangle

Such a product of bisets appears naturally out of the following topological data. Let us denote by $(S_i)_{i \in I}$ the connected components of the disconnection $(S^2, A) \setminus \mathcal{C}$, and by $\mathbf{Mod}(S_i)$ the pure mapping class group of S_i . Then the group $\mathbf{vMod}(S^2, A, \mathcal{C})$ is the direct product $\prod_i \mathbf{Mod}(S_i)$. For every $b \in vM(f, A, \mathcal{C})$ let $b_i: \widehat{U}_i \rightarrow \widehat{S_{f(i)}}$ be the small sphere map induced by b as in Definition 4.9 (note that all $b \in vM(f, A, \mathcal{C})$ induce the same map on I). Denote by $vM(f, A, \mathcal{C})_i$ the restriction of $vM(f, A, \mathcal{C})$ to S_i :

$$vM(f, A, \mathcal{C})_i := \mathbf{Mod}(S_i) \{b_i \mid b \in vM(f, A, \mathcal{C})\} \mathbf{Mod}(S_{f(i)}).$$

Lemma 6.19. *The $\mathbf{vMod}(S^2, A, \mathcal{C})$ -biset $vM(f, A, \mathcal{C})$ is the product*

$$vM(f, A, \mathcal{C}) = \prod_i vM(f, A, \mathcal{C})_i$$

of the $\mathbf{Mod}(S_i)$ - $\mathbf{Mod}(S_{f(i)})$ -bisets $vM(f, A, \mathcal{C})_i$. \square

Proposition 6.20. *Consider a periodic cycle*

$$(29) \quad \Pi = \{i, f(i), \dots, f^{(t)}(i) = i\}$$

of $f: I \hookrightarrow I$, and assume that an oracle solves the conjugacy and centralizer problems in the $\mathbf{vMod}(S_i)$ -biset

$$(30) \quad vM(f, A, \mathcal{C})_i \otimes_{\mathbf{vMod}(S_{f(i)})} vM(f, A, \mathcal{C})_{f(i)} \otimes \dots \otimes vM(f, A, \mathcal{C})_{f^{t-1}(i)}.$$

Then the conjugacy and centralizer problems are solvable in the $\prod_{j \in \Pi} \mathbf{vMod}(S_j)$ -biset

$$(31) \quad vM(f, A, \mathcal{C})_\Pi := \prod_{j \in \Pi} vM(f, A, \mathcal{C})_j.$$

If for every periodic cycle as in (29) the conjugacy and centralizer problems are solvable in (31), then the conjugacy and centralizer problems are solvable in $vM(f, A, \mathcal{C})$.

We will prove the proposition later, in the form of Algorithm 6.22; we start by deriving some consequences. We remark that the bisets defined by (30) are the first return maps of f on the system of small spheres $(S_i)_{i \in I}$. As a corollary, we have:

Theorem 6.21. *There is an algorithm with oracle that, GIVEN two $g_1, g_2 \in M(f, A, \mathcal{C})$ SOLVES the conjugacy problem between $g_1, g_2 \in M(f, A, \mathcal{C})$ and COMPUTES the centralizer of g_1 as the kernel of a homomorphism from a finite-index subgroup of a product of mapping class groups towards a finitely generated abelian group, PROVIDED THAT the oracle solves conjugacy problems between all small Thurston maps of g_1 and the associated small Thurston of g_2 and PROVIDED THAT the oracle computes the centralizer of all small Thurston maps of f .*

Proof. This follows directly from Propositions 6.17 and 6.20. \square

We recall again that $g_1, g_2 \in M(f, A, \mathcal{C})$ are typically given in terms of their bisets. By Theorem 3.9 the sphere tree of bisets decomposing $B(g_1)$ and $B(g_2)$ relatively to \mathcal{C} are computable; so the projections of g_1 and g_2 in $vM(f, A, \mathcal{C})$, see (24), and the bisets of small Thurston maps of g_1 and g_2 , see (12), are computable.

Note that the centralizer of a homeomorphism is usually not very complicated. For example, if $h: (S^2, A, \mathcal{C}) \hookrightarrow (S^2, A, \mathcal{C})$ is a homeomorphism, then its centralizer is, up to

finite index, the direct product of the centralizers of the small Thurston maps. The situation is much more intricate for non-invertible maps, see §8.3.

Algorithm 6.22. GIVEN a multicurve \mathcal{C} and a solution, for each periodic cycle of small spheres, of the conjugacy and centralizer problems in the first return biset, SOLVE the conjugacy and centralizer problems in $vM(f, A, \mathcal{C})$ AS FOLLOWS:

Consider two elements $b, c \in vM(f, A, \mathcal{C})$ for which we wish to know whether they are conjugate and for which we wish to compute $Z(b)$.

- (1) We use the notation b_i for the projection of $b \in vM(f, A, \mathcal{C})$ to $vM(f, A, \mathcal{C})_i$, etc.

Let $\Pi_1, \Pi_2, \dots, \Pi_n$ be all the periodic cycles of $f: I \hookrightarrow$, viewed as subsets of I . To avoid hard-to-read subscripts, we introduce the notation $b[\Pi_1], \dots, b[\Pi_n]$ for the images of b in $vM(f, A, \mathcal{C})_{\Pi_1}, \dots, vM(f, A, \mathcal{C})_{\Pi_n}$ respectively, and define $c[\Pi_1], \dots, c[\Pi_n]$ similarly.

- (2) If for some cycle Π_j the element $b[\Pi_j]$ is not conjugate in $vM(f, A, \mathcal{C})_{\Pi_j}$ to $c[\Pi_j]$, then b and c are not conjugate, and we have solved the conjugacy problem for b, c in the negative.
- (3) For every $X \subset I$ with $f(X) \subset X$, let us denote by

$$\mathbf{Mod}_X := \prod_{i \in X} \mathbf{Mod}(\hat{S}_i)$$

the pure mapping class group of $\bigsqcup_{i \in X} \hat{S}_i$, by

$$vM(f, A, \mathcal{C})_X := \prod_{i \in X} vM(f, A, \mathcal{C})_i$$

the mapping class biset over $\bigsqcup_{i \in X} \hat{S}_i$, and by $b[X]$ and $c[X]$ the images of b and c in $vM(f, A, \mathcal{C})_X$ respectively. We denote also by $Z(b[X])$ the centralizer of $b[X]$ in $vM(f, A, \mathcal{C})_X$, and by $h[X]$ an element of \mathbf{Mod}_X that conjugates $b[X]$ to $c[X]$, namely satisfies $b[X] = c[X]^{h[X]}$; or, if $b[X], c[X]$ are not conjugate, we write $h[X] = \mathbf{fail}$.

- (4) Let $X_0 = \bigsqcup_j \Pi_j$ be the union of the cycles in I . From the assumptions of the proposition, $h[X_0]$ and $Z(b[X_0])$ are computable, and we show now how to compute the variables $h[X]$ and $Z(b[X])$ for all $X \subseteq I$.

After all $h[X]$ and $Z(b[X])$ have been computed, we will in particular know whether b and c are conjugate: if $h[I] \neq \mathbf{fail}$, then it conjugates b into c . The centralizer of b in $Z(b[I])$.

- (5) Assume that $h[X]$ and $Z(b[X])$ have been computed, and consider $i \in I \setminus X$ with $f(i) \in X$. This is how to compute $h[X \cup \{i\}]$ and $Z(b[X \cup \{i\}])$.

If $h[X] = \mathbf{fail}$ then set $h[X \cup \{i\}] := \mathbf{fail}$. Otherwise, compute the action of $Z(b[X])[f(i)] \leq \mathbf{Mod}(\hat{S}_{f(i)})$ on $1 \otimes c_i h[f(i)] \in 1 \otimes vM(f, A, \mathcal{C})_i$. Since $Z(b[X])$ is computable and in particular finitely generated and it acts on a finite set, the orbit of $1 \otimes c_i h[f(i)]$ is computable. If this orbit does not contain $1 \otimes b_i$, then set $h[X \cup \{i\}] := \mathbf{fail}$. Otherwise, let $m \in Z(b[f(i)])$ and $h[i] \in \mathbf{Mod}(\hat{S}_i)$ be such that $h[i]b_i = c_i h[f(i)]m[f(i)]$; by replacing $h[X]$ with $h[X]m[X]$ and defining $h[i]$ as above, we extend $h[X]$ to $h[X \cup \{i\}]$.

Again we use the fact that the action of $Z(b[X])$ on the finite set $1 \otimes vM(f, A, \mathcal{C})_i$ is computable; let H be the finite-index subgroup of $Z(b[X])$ that stabilizes $1 \otimes b_i$. Furthermore, let $\phi: H \rightarrow \mathbf{Mod}(\hat{S}_i)$ satisfy $a^\phi b_i =$

$b_i a[f(i)]$; it is well-defined by Proposition 6.4. Let then $Z(b[X \cup \{i\}])$ be the diagonal image $a \mapsto (a, a^\phi)$ of H in $Z(b[X]) \cup \mathbf{Mod}(\widehat{S}_i) \leq \mathbf{Mod}_{X \cup \{i\}}$.

Proof of Proposition 6.20. The second part is proven in Algorithm 6.22.

For the first part, consider a new set $\Pi' = \{i, f(i), \dots, f^{(t-1)}(i), f^{(t)}(i) = i' = f(i')\}$. Set $S_{i'} := S_i$ and extend f to $S_{i'}$ as $f^{(t)}$. Then the biset $vM(f, A, \mathcal{C})_{i'}$ is given by (30), which by assumption has solvable conjugacy and centralizer problems. Since this is the only cycle of Π' , we may apply the second part and deduce that the conjugacy and centralizer problems are solvable for Π' .

Consider now two elements $b, c \in vM(f, A, \mathcal{C})_\Pi$ for which we wish to know whether they are conjugate and for which we wish to compute $Z(b)$.

Write $b = (b_i, b_{f(i)}, \dots, b_{f^{(t-1)}(i)})$ and $c = (c_i, c_{f(i)}, \dots, c_{f^{(t-1)}(i)})$. Extend them to elements $b', c' \in vM(f, A, \mathcal{C})_{\Pi'}$ by setting $b_{i'} = b_i \otimes \dots \otimes b_{f^{(t-1)}(i)}$ and $c_{i'} = c_i \otimes \dots \otimes c_{f^{(t-1)}(i)}$.

Then b, c are conjugate if and only if b', c' are conjugate in $vM(f, A, \mathcal{C})_{\Pi'}$. If $h' = (h_i, h_{f(i)}, \dots, h_{i'})$ conjugates b' to c' , then $h = (h_i, \dots, h_{f^{(t-1)}(i)})$ conjugates b to c . The centralizer $Z(b)$ coincides with the centralizer $Z(b')$. Therefore, both problems are solvable. \square

7. ORBISPHERES

All the previous considerations, on marked spheres, apply equally well to a slightly more general situation, that of *orbispheres*. Consider a marked sphere (S^2, A) , and let there also be given a function $\text{ord}: A \rightarrow \{2, 3, \dots, \infty\}$, assigning a positive or infinite degree to each marked point. This describes an *orbispace* structure: if $\text{ord}(a) = \infty$, then the sphere is punctured at $a \in A$, while if $\text{ord}(a) = n$ then the space has a cone-type singularity of angle $2\pi/n$ at a . This can be thought of as specifying a complex structure on S^2 : a small enough neighbourhood of a point $a \in A$ with $\text{ord}(a) = \infty$ is identified with a neighbourhood of 0 in \mathbb{C}^* ; while if $\text{ord}(a) \in \mathbb{N}$, a neighbourhood of a is modelled on a neighbourhood of 0 in $\mathbb{C}/\langle e^{2\pi i/\text{ord}(a)} \rangle$. The parabolic class Γ_i consists in elements of order $\text{ord}(\Gamma_i) := \text{ord}(a_i)$, see (32). For more details, consult [26, Appendix A].

Orbisppheres are denoted (S^2, A, ord) , or just (S^2, A) if the degree function is implicit. To avoid degenerate cases, assume $\#A \neq 1$; and if $\#A = 2$, assume furthermore that ord is constant on A . The Euler characteristic of (S^2, A, ord) is defined as

$$\chi(S^2, A, \text{ord}) = 2 - \sum_{a \in A} \left(1 - \frac{1}{\text{ord}(a)}\right).$$

Accordingly, (S^2, A, ord) is called *spherical*, *euclidean* or *hyperbolic* if its Euler characteristic is > 0 , $= 0$ or < 0 . If we endow S^2 with a complex structure, then the *universal cover* $(\widetilde{S^2, A, \text{ord}})$ of (S^2, A, ord) is respectively $\widehat{\mathbb{C}}$, \mathbb{C} or $\mathbb{D}(0, 1)$ with its usual complex structure [26, Theorem A.2]; in the sense that $(S^2, A, \text{ord}) = (\widetilde{S^2, A, \text{ord}})/G$ for a group G of isometries acting with finite stabilizers. Those points with non-trivial stabilizers project then precisely to the marked points A ; because of the ambient complex structure, these stabilizers are perforce cyclic groups, and the order of $G_{\bar{a}}$ is $\text{ord}(a)$.

The same loops γ_i as before generate G ; but now, $\gamma_i^{\text{ord}(a_i)} = 1$ whenever $\text{ord}(a_i) < \infty$. We have

$$(32) \quad G = \pi_1(S^2, A, \text{ord}, *) = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1^{\text{ord}(a_1)}, \dots, \gamma_n^{\text{ord}(a_n)}, \gamma_1 \cdots \gamma_n \rangle.$$

Indeed G is a group of isometries of (S^2, A, ord) , with fundamental domain a $\#A$ -sided polygon with angles $2\pi/\text{ord}(a_i)$; so the presentation (32) follows from Poincaré's Theorem, see e.g. [17, Theorem V.B.40].

Definition 7.1 (Orbisphere groups). An *orbisphere group* is a tuple $(G, \Gamma_1, \dots, \Gamma_n)$ consisting of a group and $n \neq 1$ conjugacy classes Γ_i in G , such that G admits a presentation as in (32) for some choice of $\gamma_i \in \Gamma_i$. The *Euler characteristic* of G is

$$\chi(G) = 2 - \sum_{i=1}^n \left(1 - \frac{1}{\text{ord}(a_i)}\right).$$

Suppose that $G_1 = \pi_1(S^2, A, \text{ord}_1, *)$ and $G_2 = \pi_1(S^2, A, \text{ord}_2, *)$ are two orbisphere groups and suppose that $\text{ord}_2(a) \mid \text{ord}_1(a)$ for all $a \in A$. Then the natural homomorphism $G_1 \rightarrow G_2$ is called an *inessential forgetful map*. \triangle

It is convenient to extend 'ord' to S^2 so that $\text{ord}(p) = 1 \Leftrightarrow p \notin A$. Note again that marked spheres are subsumed in the definition of orbispheres; namely, as those for which $\text{ord}(a) = \infty$ for all $a \in A$.

7.1. Pure homeomorphisms between orbispheres.

Definition 7.2 (Orbisphere maps). An *orbisphere map* $f: (S^2, C, \text{ord}_C) \rightarrow (S^2, A, \text{ord}_A)$ between orbispheres is a branched covering between the underlying spheres, with $f(C) \cup \{\text{critical values of } f\} \subseteq A$, that is locally modelled at $p \in S^2$ in oriented complex charts by $z \mapsto z^{\deg_p(f)}$ for some integer $\deg_p(f) \geq 1$, so that $\text{ord}_C(p) \deg_p(f) \mid \text{ord}_A(f(p))$ for all $p \in S^2$. A *covering* between orbispheres is an orbisphere map f for which one has $\text{ord}_C(p) \deg_p(f) = \text{ord}_A(f(p))$ for all $p \in S^2$.

An *anti-orbisphere map* $f: (S^2, C, \text{ord}_C) \rightarrow (S^2, A, \text{ord}_A)$ between orbispheres is a branched covering satisfying the same condition as above except that f is locally modelled in oriented charts at $p \in S^2$ by $z \mapsto \bar{z}^{\deg_p(f)}$ for some integer $\deg_p(f) \geq 1$. \triangle

In particular, if f is a covering then $A = f(C \cup \{\text{critical points of } f\})$.

For an orbisphere (S^2, A, ord) we define the *modular group* consisting of orbisphere and anti-orbisphere pure homeomorphisms as

$$\mathbf{Mod}^\pm(S^2, A, \text{ord}) = \{h: (S^2, A, \text{ord}) \hookrightarrow \mid \deg(h) = 1, h|_A = \mathbb{1}\},$$

and we denote by $\mathbf{Mod}(S^2, A, \text{ord})$ the subgroup of order-preserving homeomorphisms in $\mathbf{Mod}^\pm(S^2, A, \text{ord})$. Clearly, $\mathbf{Mod}(S^2, A, \text{ord})$ and $\mathbf{Mod}(S^2, A)$ are isomorphic.

Choose $* \in S^2 \setminus A$, write $G = \pi_1(S^2, A, \text{ord}, *)$, and set

$$\mathbf{Mod}^\pm(G) = \{\phi \in \text{Out}(G) \mid \Gamma_i^\phi = \Gamma_i^{\pm 1} \forall i = 1, \dots, n\}.$$

We have the following generalization of Theorem 2.3.

Theorem 7.3 (Dehn-Nielsen-Baer-Zieschang-Vogt-Coldewey). *Let (S^2, A, ord) be an orbisphere with non-positive Euler characteristic. Then the natural map $\mathbf{Mod}^\pm(S^2, A, \text{ord}) \rightarrow \mathbf{Mod}^\pm(\pi_1(S^2, A, \text{ord}))$ is an isomorphism.*

Proof. This is essentially a direct consequence of [33, Theorems 5.8.3 and 5.14.1]. Since these theorems deal with groups that have only “finite order peripheral classes” (such a set is always preserved by automorphisms), we need to make slight adjustments.

Let us assume $\#A \geq 4$; if not, $\mathbf{Mod}^\pm(S^2, A, \text{ord})$ is a group of order two and the claim is easy to verify.

Define a new map $\text{ord}' : A \rightarrow \{2, 3, \dots\}$ by

$$\text{ord}'(a) = \begin{cases} \text{ord}(a) & \text{if } \text{ord}(a) < \infty, \\ 2016 & \text{if } \text{ord}(a) = \infty. \end{cases}$$

We still have $\chi(S^2, A, \text{ord}) \leq 0$ because $\#A \geq 4$. Since the natural map $\mathbf{Mod}^\pm(S^2, A, \text{ord}) = \mathbf{Mod}^\pm(S^2, A, \text{ord}') \rightarrow \mathbf{Mod}^\pm(\pi_1(S^2, A, \text{ord}'))$ factors as

$$\mathbf{Mod}^\pm(S^2, A, \text{ord}) \rightarrow \mathbf{Mod}^\pm(\pi_1(S^2, A, \text{ord})) \rightarrow \mathbf{Mod}^\pm(\pi_1(S^2, A, \text{ord}')),$$

it suffices to prove the theorem for the orbisphere (S^2, A, ord') .

Consider $\phi \in \mathbf{Mod}^\pm(\pi_1(S^2, A, \text{ord}'))$. By [33, Theorem 5.8.3] there is a homeomorphism $h_\phi : (S^2 \setminus A) \hookrightarrow$ inducing ϕ . By [33, Theorem 5.14.1] it is unique up to isotopy rel A . Indeed, if h'_ϕ is another such homeomorphism, then the lift of $h_\phi \circ h_\phi^{-1}$ to the universal cover $\pi_1(S^2, A, \text{ord}')U$ commutes with the action of $\pi_1(S^2, A, \text{ord}')$; and by [33, Theorem 5.14.1] $h_\phi \circ h_\phi^{-1}$ is isotopic to the identity relatively to the action of $\pi_1(S^2, A, \text{ord}')$. We get a homomorphism $\mathbf{Mod}^\pm(\pi_1(S^2, A, \text{ord}')) \rightarrow \mathbf{Mod}^\pm(S^2, A, \text{ord}')$ that is inverse to $\mathbf{Mod}^\pm(S^2, A, \text{ord}') \rightarrow \mathbf{Mod}(\pi_1(S^2, A, \text{ord}'))$. \square

For $G = \pi_1(S^2, A, \text{ord})$, define $\mathbf{Mod}(G)$ to be the image in $\mathbf{Mod}^\pm(G)$ of $\mathbf{Mod}(S^2, A, \text{ord})$.

Corollary 7.4. *Let G be an orbisphere group and let \tilde{G} be a sphere group together with an inessential forgetful map $\tilde{G} \rightarrow G$. Then the natural map $\tilde{G} \rightarrow G$ induces an isomorphism $\mathbf{Mod}(\tilde{G}) \rightarrow \mathbf{Mod}(G)$.*

Proof. If $\chi(G) > 0$, then $\mathbf{Mod}(\tilde{G})$ and $\mathbf{Mod}(G)$ are trivial groups. The case $\chi(G) \leq 0$ follows from Theorem 7.3. \square

Since $\mathbf{Mod}(G)$ is countable and its elements can be explicitly enumerated we also have

Corollary 7.5. *Let $\tilde{G} \rightarrow G$ be as in Corollary 7.4. There is an algorithm that, given $h \in \mathbf{Mod}(G)$ computes its preimage under the forgetful map $\mathbf{Mod}(\tilde{G}) \rightarrow \mathbf{Mod}(G)$.* \square

Remark 7.6. A more effective algorithm is to decompose $h \in \mathbf{Mod}(G)$ into a product of Dehn twists, and then to lift each Dehn twist to $\mathbf{Mod}(\tilde{G})$. This will be developed in [6].

Proposition 7.7. *Let G be an orbisphere group.*

If $\chi(G) \leq 0$, then $\mathbf{Mod}(G)$ has index two in $\mathbf{Mod}^\pm(G)$, and there is an algorithm that, given $\phi \in \mathbf{Mod}^\pm(G)$, determines whether ϕ belongs to $\mathbf{Mod}(G)$.

If G has at least one peripheral conjugacy class Γ with $\text{ord}(\Gamma) > 2$, then $\phi \in \mathbf{Mod}(G)$ if and only if $\Gamma^\phi = \Gamma$.

Proof. There exist anti-orbisphere maps, so $\mathbf{Mod}(G) \neq \mathbf{Mod}^\pm(G)$. This can be seen algebraically, by considering the map ϕ given on generators $\gamma_1, \dots, \gamma_n$ with $\gamma_1 \cdots \gamma_n = 1$ by $\gamma_i^\phi = \gamma_i^{-\gamma_{i-1}^{-1} \cdots \gamma_1^{-1}}$. By Theorem 7.3 we have $[\mathbf{Mod}^\pm(G) : \mathbf{Mod}(G)] = 2$ if $\chi(G) \leq 0$.

If ϕ is induced by an anti-orbisphere map, then $\phi(\Gamma) = \Gamma^{-1} \neq \Gamma$; this proves the last claim, and also gives an algorithm in case at least one peripheral conjugacy class has order > 2 .

Suppose now that all peripheral conjugacy classes in G have order 2, and let us give an algorithm deciding whether $\phi \in \mathbf{Mod}^\pm(G)$ is in $\mathbf{Mod}(G)$. Denote by Γ_i all peripheral conjugacy classes in G , so G has presentation $\langle \gamma_1, \dots, \gamma_n \mid \gamma_i^2, \gamma_1 \cdots \gamma_n \rangle$. Consider the map $G \rightarrow \mathbb{Z}/2$ mapping generators $\gamma_1, \dots, \gamma_4$ to 1 (mod 2) and the others to 0 (mod 2), and let H denote the kernel of this map. Setting $\alpha = \gamma_2 \gamma_1$ and $\beta = \gamma_1 \gamma_3$ and $\delta_i = \gamma_i^{\gamma_1 \gamma_2 \gamma_3}$ for $n = 5, \dots, n$, we obtain by the Reidemeister-Schreier theorem (see e.g. [24, §2.3]) the presentation

$$H = \langle \alpha, \beta, \gamma_5, \delta_5, \dots, \gamma_n, \delta_n \mid [\alpha, \beta] \delta_5 \cdots \delta_n \gamma_5 \cdots \gamma_n, \gamma_i^2, \delta_i^2 \rangle.$$

Consider $\phi \in \mathbf{Mod}^\pm(G)$. Since ϕ preserves every Γ_i , it also preserves H , and therefore induces an automorphism ϕ_* of $H/[H, H] \otimes \mathbb{Q} \cong \mathbb{Q}^2$, whose action on generators can be explicitly computed. We have

$$\mathbf{Mod}(G) = \{\psi \in \mathbf{Mod}^\pm(G) \mid \det(\psi_*) = 1\},$$

so to determine whether $\phi \in \mathbf{Mod}(G)$ it suffices to compute the action of ϕ_* on $H/[H, H] \otimes \mathbb{Q}$. Note that, for the orientation-reversing map ϕ given at the beginning of the proof, we have $\phi_* = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ in basis $\{\alpha, \beta\}$. \square

Remark 7.8. In the above proof, the homomorphism $G \twoheadrightarrow \mathbb{Z}/2$ constructs a genus-1 cover of the punctured sphere; it is on the cover that the orientation of a mapping class can be read. More conceptually, if n is even then we may consider $H = \ker(\gamma_i \mapsto 1 \pmod{2})$, corresponding to the hyperelliptic cover of the n -punctured sphere, which is then a surface of genus $n/2 - 1$ without punctures; then ϕ_* is the action of ϕ on the homology of this cover.

7.2. Orbisphere bisets. Let H, G be orbisphere groups. An *orbisphere H - G -biset* B is defined exactly as in Definition 2.6 — the only difference is that the acting groups are orbisphere groups. As in Definition 2.6 the biset B induces a map $B_* : C \rightarrow A$ from peripheral conjugacy classes of G indexed as $(\Gamma_a)_{a \in A}$ to peripheral conjugacy classes of H indexed as $(\Delta_c)_{c \in C}$. In the dynamical case $G = H$, then we have a self-map $B_* : A \hookrightarrow A$ called the *portrait* of B .

The same argument as in Lemma 2.7 shows that the biset of an orbisphere map is an orbisphere biset. Moreover, if $\text{ord}_C(C) = \{2\}$, then the biset of an anti-orbisphere map is an orbisphere biset. We are now ready to generalize Theorem 2.8. We omit the case $\text{ord}_C(C) = \{2\}$ and $\chi(G) > 0$ from the theorem — this case will be illustrated in Example 7.13.

Theorem 7.9. *Suppose that (S^2, C, ord_C) and (S^2, A, ord_A) are marked orbispheres with $\#C \geq 2$; write $H := \pi_1(S^2, C, \text{ord}_C, \dagger)$ and $G := \pi_1(S^2, A, \text{ord}_A, *)$ for choices of $\dagger \in S^2 \setminus C$ and $*$ in $S^2 \setminus A$.*

Suppose first $\text{ord}_C(C) \neq \{2\}$. Then orbisphere maps $f_0, f_1: (S^2, C, \text{ord}_C) \rightarrow (S^2, A, \text{ord}_A)$ are isotopic if and only if $B(f_0) \cong B(f_1)$. Conversely, for every orbisphere H - G -biset B there exists an orbisphere map $f: (S^2, C) \rightarrow (S^2, A)$, unique up to isotopy, such that $B \cong B(f)$.

Suppose that $\text{ord}_C(C) = \{2\}$ but $\chi(G) \leq 0$. Then orbisphere or anti-orbisphere maps $f_0, f_1: (S^2, C, \text{ord}_C) \rightarrow (S^2, A, \text{ord}_A)$ are isotopic if and only if $B(f_0) \cong B(f_1)$. Conversely, for every orbisphere H - G -biset B there exists an orbisphere or anti-orbisphere map $f: (S^2, C) \rightarrow (S^2, A)$, unique up to isotopy, such that $B \cong B(f)$.

The proof is essentially the same as that of Theorem 2.8; therefore we only sketch the argument, underlining the differences.

Proof. Recall that the proof of Theorem 2.8 is based on Theorem 2.3, on Decomposition (6) and on Lemmas 2.9 and 2.10. Theorem 2.3 is generalized by Theorem 7.3 — see Corollary 7.4. Decomposition (6) holds for orbisphere bisets, while Lemma 2.9 is adjusted as follows, with a completely analogous proof:

Lemma 7.10. *Suppose that ${}_H B_G$ is the biset of an orbisphere map $f: (S^2, C, \text{ord}_C) \rightarrow (S^2, A, \text{ord}_A)$ with $G = \pi_1(S^2 \setminus A, *)$ and $H = \pi_1(S^2 \setminus C, \dagger)$. If $\text{ord}_C(C) = \{2\}$, then allow f to be an anti-orbisphere map.*

For $p \in S^2$ define $\text{ord}_{\tilde{A}}(p) := \text{ord}_A(f(p)) / \deg_p(f)$ and set $\tilde{A} := \{p \in S^2 \mid \text{ord}_{\tilde{A}}(p) > 1\}$ so that $f: (S^2, \tilde{A}, \text{ord}_{\tilde{A}}) \rightarrow (S^2, A, \text{ord}_A)$ is a covering of orbispheres. Consider $b \in B$ and let $*'$ be the endpoint of b .

Then $\pi_1(S^2, \tilde{A}, \text{ord}_{\tilde{A}}, *')$ is identified via $f_*: \pi_1(S^2, \tilde{A}, \text{ord}_{\tilde{A}}, *') \rightarrow \pi_1(S^2, A, *)$ with G_b , and via this identification the $\pi_1(S^2, \tilde{A}, \text{ord}_{\tilde{A}}, *')$ - G -biset of $f: (S^2, f^{-1}(A)) \rightarrow (S^2, A)$ is isomorphic to ${}_{G_b} G_G$ while the H - $\pi_1(S^2, f^{-1}(A), *')$ -biset of $(S^2, C) \xrightarrow{1} (S^2, f^{-1}(A))$ is isomorphic to $H b G_b$.

Moreover, via the identification of G_b with $\pi_1(S^2, \tilde{A}, \text{ord}_{\tilde{A}}, *')$ the peripheral conjugacy classes of G_b are $(\Xi_{i,j})_{i,j}$ constructed as follows. Let $\Gamma_1, \dots, \Gamma_n$ be the peripheral conjugacy classes of G . Then for each Γ_i there is a unique decomposition

$$(33) \quad (\Gamma_i^+ \setminus \{1\}) \cap G_b = (\Xi_{i,1}^+ \setminus \{1\}) \sqcup (\Xi_{i,2}^+ \setminus \{1\}) \sqcup \dots \sqcup (\Xi_{i,s}^+ \setminus \{1\})$$

such that every $\Xi_{i,j}$ is a non-trivial conjugacy class of G_b . Assuming $\Xi_{i,j}$ is generated by $\gamma_{i,j}^{d(i,j)}$ with $\gamma_{i,j} \in \Gamma_i$ and with minimal possible $d(i,j) \geq 1$, we let $\{(d(i,j), \Xi_{i,j})\}$ be the multiset of lifts of Γ_i via ${}_{G_b} G_G$. \square

Lemma 2.10, as well as its proof, holds for orbisphere bisets, as long as references to Lemma 2.9 are replaced by references to Lemma 7.10.

Now the proof copies that of Theorem 2.8 with the exception that in the case $\text{ord}_C(C) = \{2\}$ anti-orbisphere maps are indistinguishable from orbisphere maps. \square

Corollary 7.11. *Let $\tilde{G} \rightarrow G$ and $\tilde{H} \rightarrow H$ be as in Corollary 7.4. There is an algorithm that, given B an H - G -biset, computes its preimage under the forgetful map.*

Proof. Anti-sphere bisets are defined in complete analogy with sphere bisets, see Definition 2.6, except that the multisets of all lifts of Γ_i contain Δ_j^{-1} instead of Δ_j . Enumerate then all sphere and anti-sphere \tilde{H} - \tilde{G} -bisets \tilde{B} , and return the first one that maps to B under the inessential forgetful maps. \square

7.2.1. Orbisphere bisets and orientation. By Lemma 2.2, to every non-trivial sphere group there is a uniquely associated oriented sphere. This is also true for orbisphere groups, except if all peripheral conjugacy classes have order 2; in that case, it is not possible to recover the orientation of the topological orbisphere: if $A \subset \mathbb{R}$, the map $z \mapsto \bar{z}$: $(\hat{\mathbb{C}}, A, \text{ord}) \hookrightarrow$ has biset isomorphic to that of $z \mapsto z$. Thus, in Corollary 7.11, the choices of \tilde{G} and \tilde{H} amount to choices of orientations above the orbispheres of G, H respectively.

Proposition 7.12. *Suppose that in an orbisphere group H all peripheral conjugacy classes have order 2 and $\chi(H) \leq 0$.*

Then there is an algorithm that decides whether the bisets ${}_HB_G, {}_HC_G$ have the same or different orientation. In particular, there is an algorithm deciding whether an orbisphere biset ${}_HB_H$ corresponds to an orientation preserving map.

Proof. Choose by Corollary 7.5 sphere groups \tilde{G}, \tilde{H} above G, H respectively. Using Corollary 7.11, find \tilde{H} - \tilde{G} -biset \tilde{B}, \tilde{C} above B, C respectively. Then B, C have the same orientation if and only if \tilde{B}, \tilde{C} are either both sphere or both anti-sphere bisets.

The second statement follows from the first, by taking $G = H$ and $C = {}_GG_G$. \square

Note that if H has four peripheral classes, so $\chi(H) = 0$, there is a more efficient algorithm: consider indeed an orbisphere biset ${}_HB_H$. Since H has a unique subgroup of index 2 isomorphic to \mathbb{Z}^2 , the restriction of B to \mathbb{Z}^2 yields a 2×2 -integer matrix M_B , and B is orientation-preserving precisely when $\det(M_B) > 0$. Details will be given in [4].

Example 7.13 (Different orbisphere bisets of the same map). Consider the following post-critically finite rational map $f(z) = \left(\frac{z^2-1}{z^2+1}\right)^2$. It has order-2 critical points at $\pm i, \pm 1, 0, \infty$ with $f(\pm i) = \infty, f(\pm 1) = 0, f(0) = f(\infty) = 1$. Let $A := \{0, 1, \infty\}$ be the post-critical set of f and write $G := \pi_1(\hat{\mathbb{C}}, A, *) = \langle a, b, c \mid abc \rangle$ so that a and c are counterclockwise loops around 0 and ∞ respectively intersecting once $\mathbb{R}_{\geq 0}$ while b is a counterclockwise loop around 1 intersecting twice $\mathbb{R}_{\geq 0}$.

We compute $B(f)$ viewed as a sphere map. Suppose $* \notin \mathbb{R}_{\geq 0}$ and let $X := \{1, 2, 3, 4\}$ be the basis of $B(f)$ normalized so that 1, 2, 3, 4 are unique curves in $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ connecting $*$ to its unique preimage in $\{z \mid \text{Re}(z) > 0, \text{Im}(z) > 0\}, \{z \mid \text{Re}(z) > 0, \text{Im}(z) < 0\}, \{z \mid \text{Re}(z) < 0, \text{Im}(z) < 0\}, \{z \mid \text{Re}(z) < 0, \text{Im}(z) > 0\}$ respectively. Then $B(f)$ is described by the recursion

$$\begin{aligned} (34) \quad & a \mapsto \langle a^{-1}, 1, 1, ab \rangle (1, 4)(2, 3) \\ & b \mapsto \langle c, 1, 1, a \rangle (1, 3)(2, 4) \\ & c \mapsto \langle 1, 1, 1, 1 \rangle (1, 2)(3, 4). \end{aligned}$$

View next f as an orbisphere map $\hat{\mathbb{C}} \rightarrow (\hat{\mathbb{C}}, A, \text{ord})$ with $\text{ord}(A) = \{2\}$. Then $f: \hat{\mathbb{C}} \rightarrow (\hat{\mathbb{C}}, A, \text{ord})$ is a covering of orbifolds with $\hat{\mathbb{C}} / \langle -z, \frac{1}{z} \rangle \xrightarrow{f} (\hat{\mathbb{C}}, A, \text{ord})$. Write $G' := \pi_1(\hat{\mathbb{C}}, A, \text{ord}, *) = \langle a, b, c \mid abc, a^2, b^2, c^2 \rangle \cong \langle -z, \frac{1}{z} \rangle$. Then G' is isomorphic to $(\mathbb{Z}/2)^2$ and the biset of $f: \hat{\mathbb{C}} \rightarrow (\hat{\mathbb{C}}, A, \text{ord})$ is

$$1 \otimes_G B(f) \otimes_G G' \cong 1(\mathbb{Z}/2)^2_{(\mathbb{Z}/2)^2}.$$

Finally view f as an orbisphere map $(\widehat{\mathbb{C}}, A, \text{ord}) \rightarrow (\widehat{\mathbb{C}}, A, \text{ord}^\circ)$ with $\text{ord}^\circ(a) = \text{ord}^\circ(b) = 4$ and $\text{ord}^\circ(c) = 2$. Write $G^\circ := \pi_1(\widehat{\mathbb{C}}, A, \text{ord}^\circ, *) = \langle a, b, c \mid abc, a^4, b^4, c^2 \rangle$. Then $\chi(G^\circ) = 0$ and G° has an index 3-subgroup isomorphic to \mathbb{Z}^2 . The biset of $(\widehat{\mathbb{C}}, A, \text{ord}) \rightarrow (\widehat{\mathbb{C}}, A, \text{ord}^\circ)$ is

$${}_{G'}B_{G^\circ}^\circ = G' \otimes_G B(f) \otimes_G G^\circ;$$

as a set it has $16 = \#G' \cdot \#X$ elements transitively permuted by G° . We note that ${}_{G'}B_{G^\circ}^\circ$ uniquely determines f among all (orientation preserving) orbisphere maps from $(\widehat{\mathbb{C}}, A, \text{ord})$ to $(\widehat{\mathbb{C}}, A, \text{ord}^\circ)$; however the biset of anti-orbisphere map $f \circ \bar{z}: (\widehat{\mathbb{C}}, A, \text{ord}) \rightarrow (\widehat{\mathbb{C}}, A, \text{ord}^\circ)$ is also ${}_{G'}B_{G^\circ}^\circ$.

7.3. Inessential forgetful morphisms. Consider a Thurston map $f: (S^2, A) \hookrightarrow$. There exist various orbispace structures on (S^2, A) , namely functions $\text{ord}: A \rightarrow \{2, 3, \dots, \infty\}$, that turn f into an orbisphere map. The minimal one is given by

$$(35) \quad \text{ord}_f(a) = \text{l. c. m.} \{ \deg_z(f^k) \mid k \in \mathbb{N}, z \in f^{-k}(a) \} \in \mathbb{N} \cup \{\infty\},$$

and is denoted by (S^2, P_f, \deg_f) with $P_f \subseteq A$ the post-critical set of f . The maximal one is given by $\deg_\infty(a) = \infty$ for all $a \in A$, and is denoted by (S^2, A, \deg_∞) . The former is the usual one to consider when one is just given a map f ; the latter has the advantage of being defined on a punctured sphere rather than on an orbispace; but intermediate orbispace structures will be required, firstly because sometimes the map f to consider is given by a previous construction that specified its orbispace structure; secondly because periodic non-postcritical cycles are invisible in the fundamental group of the minimal orbispace structure (they correspond to generators of order 1); and thirdly because the maximal structure will not satisfy the required “contraction” properties in the next article [4] of the series. If periodic cycles were marked in A , then \deg_f can be extended to $A \setminus P_f$ by taking a constant value, for example 2, on non-post-critical points.

These notions have algebraic counterparts. Consider an orbisphere biset ${}_GB_G$. It has a *portrait* $B_*: A \hookrightarrow$ induced by the map on peripheral conjugacy classes, and a *local degree* $\deg_a(B)$. There is a *minimal orbisphere quotient* of G associated with B , which is the quotient \overline{G} of G by the additional relations $\Gamma_a^{\text{ord}_B(a)} = 1$, for

$$(36) \quad \text{ord}_B(a) = \text{l. c. m.} \{ d \mid n \geq 0 \text{ and } \Gamma_a \text{ has a lift of degree } d \text{ under } B^{\otimes n} \},$$

and clearly $\text{ord}_B(a) \mid \text{ord}_G(a)$. We call the quotient biset $\overline{G} \otimes_G B \otimes_G \overline{G}$ the *minimal orbisphere biset* of ${}_GB_G$.

There is an even smaller quotient of G than \overline{G} , informally “the smallest for which a quotient biset of B can be defined”. Let ${}_GB_G$ be an orbisphere biset; we then naturally get a right action of G on

$$T(B) := \bigsqcup_{n \geq 0} \{ \cdot \} \otimes_G B^{\otimes n}.$$

If B is left-free of degree d then $T(B)$ naturally has the structure of a d -regular rooted tree: if S is a basis of B , then $T(B)$ is in bijection with the set of words S^* , which forms a $\#S$ -regular tree if one puts an edge between $s_1 \dots s_n$ and $s_1 \dots s_{n+1}$ for all $s_i \in S$. The action of G on $T(B)$ need not be free; following [27, 5.1.1] we denote by $\text{IMG}_B(G)$ the quotient of G by the kernel of this action.

Let \tilde{G} and G be two orbisphere groups with a forgetful morphism $\iota: \tilde{G} \rightarrow G$ between them. Let $\tilde{G} \tilde{B}_{\tilde{G}}$ be an orbisphere biset. If the kernel of $\iota: \tilde{G} \rightarrow G$ is

contained in the kernel of $\tilde{G} \rightarrow \text{IMG}_{\tilde{B}}(\tilde{G})$, then the biset

$$(37) \quad B := G \otimes_{\tilde{G}} \tilde{B}_{\tilde{G}} \otimes G$$

is an orbisphere G - G -biset. It is easy to see that the kernel of $G \rightarrow \overline{G}$ is contained in the kernel of $G \rightarrow \overline{\text{IMG}}(G)$.

Corollary 7.14. *Suppose that $\tilde{G} \rightarrow G$ is an inessential forgetful morphism such that $B := G \otimes_{\tilde{G}} \tilde{B}_{\tilde{G}} \otimes G$ as in (37) is an orbisphere biset. Then the natural map $\tilde{b} \mapsto 1 \otimes \tilde{b} \otimes 1$ induces an isomorphism between the mapping class bisets $M(\tilde{B})$ and $M(B)$.*

Proof. Follows immediately from Theorems 7.3 and 7.9. \square

7.4. Freely-oriented mapping class bisets. Consider $f: (S^2, C) \rightarrow (S^2, A)$ and as in (1) define the *freely-oriented mapping class biset*

$$M^\pm(f, C, A) = \{m'fm'' \mid m' \in \mathbf{Mod}^\pm(S^2, C), m'' \in \mathbf{Mod}^\pm(S^2, A)\}.$$

Then $M^\pm(f, C, A)$ is a *crossed product* of $\mathbb{Z}/2$ and $M(f, C, A)$ in the following way. On the level of mapping class groups we have short exact sequences

$$(38) \quad \begin{aligned} 1 &\longrightarrow \mathbf{Mod}(S^2, C) \longrightarrow \mathbf{Mod}^\pm(S^2, C) \longrightarrow \mathbb{Z}/2 \longrightarrow 1, \\ 1 &\longrightarrow \mathbf{Mod}(S^2, A) \longrightarrow \mathbf{Mod}^\pm(S^2, A) \longrightarrow \mathbb{Z}/2 \longrightarrow 1. \end{aligned}$$

In particular, we have a short exact sequence of bisets, see Definition 5.2,

$$(39) \quad \mathbf{Mod}(S^2, C)M^\pm(f, C, A)\mathbf{Mod}(S^2, A) \hookrightarrow \mathbf{Mod}^\pm(S^2, C)M^\pm(f, C, A)\mathbf{Mod}^\pm(S^2, A) \twoheadrightarrow \mathbb{Z}/2.$$

The sequences in (38) split; fix splittings $\sigma: \mathbb{Z}/2 \rightarrow \mathbf{Mod}^\pm(S^2, C)$ and $\sigma: \mathbb{Z}/2 \rightarrow \mathbf{Mod}^\pm(S^2, A)$. For example, we may choose $\sigma(1) = \bar{z}$ after identifying S^2 with $\hat{\mathbb{C}}$ and realizing C and A as subsets of \mathbb{R} . Then $\mathbb{Z}/2$ acts on $m \in \mathbf{Mod}(S^2, C)$ or $m \in \mathbf{Mod}(S^2, A)$ by $(m)^a = \sigma(a^{-1})m\sigma(a)$ and on $b \in M(f, C, A)$ by $(b)^a = \sigma(a^{-1})b\sigma(a)$. Using these splitting we have semidirect products $\mathbf{Mod}^\pm(S^2, C) = \mathbb{Z}/2 \ltimes_\sigma \mathbf{Mod}(S^2, C)$ and $\mathbf{Mod}^\pm(S^2, A) = \mathbb{Z}/2 \ltimes_\sigma \mathbf{Mod}(S^2, A)$. Define $\mathbb{Z}/2 \ltimes_\sigma M(f, C, A)$ to be the $(\mathbb{Z}/2 \ltimes_\sigma \mathbf{Mod}(S^2, C))$ - $(\mathbb{Z}/2 \ltimes_\sigma \mathbf{Mod}(S^2, A))$ -biset that is $\mathbb{Z}/2 \times M(f, C, A)$ as a set, endowed with the actions

$$(40) \quad (a', m') \cdot (a, g) \cdot (a'', m'') = (a'aa'', (m')^{aa''}(g)^{a''}m'').$$

Lemma 7.15 (Cross product for (39)). *The biset of $M^\pm(f, C, A)$ is isomorphic to the cross product $\mathbb{Z}/2 \ltimes M(f, C, A)$ by an isomorphism mapping $\sigma(a)g \in M^\pm(f, C, A)$ to $(a, g) \in \mathbb{Z}/2 \times M(f, C, A)$ with $g \in M(f, C, A)$. \square*

In the dynamical setting $(S^2, C) = (S^2, A)$ we may interpret (40) as a semidirect product of bisets. Denote by $K^\pm(S^2, A)$ the set of isotopy classes of sphere and anti-sphere maps $f: (S^2, A) \hookrightarrow$. It is naturally a semigroup under composition, and we have a short exact sequence of semigroups

$$1 \longrightarrow K(S^2, A) \longrightarrow K^\pm(S^2, A) \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

that splits. Fixing a splitting $\sigma: \mathbb{Z}/2 \rightarrow K^\pm(S^2, A)$ we get $K^\pm(S^2, A) \cong \mathbb{Z}/2 \ltimes_\sigma K(S^2, A)$. Of course $M^\pm(f, C, A)$ is isomorphic to the algebraic associated freely-oriented mapping class biset $M^\pm(B(f))$ consisting of sphere and anti-sphere bisets; thus we also have $M^\pm(B(f)) = \mathbb{Z}/2 \ltimes_\sigma M(B(f))$ once a splitting is fixed.

8. EXAMPLES

We briefly recall the application of Teichmüller theory to Thurston maps; see [2, §0.3] and [31].

The *Teichmüller space* of (S^2, A) is the set \mathcal{T}_A of complex structures on (S^2, A) , or equivalently Riemannian structures on $S^2 \setminus A$ of curvature -1 . We view it as the set of homeomorphisms $(S^2, A) \rightarrow \widehat{\mathbb{C}}$ up to isotopy rel A and post-composition by Möbius transformations. *Moduli space* \mathcal{M}_A is the set of embeddings $A \hookrightarrow \widehat{\mathbb{C}}$ up to post-composition by Möbius transformations. Teichmüller space is contractible, while the fundamental group of \mathcal{M}_A is $\mathbf{Mod}(S^2, A)$; there is a natural map $\mathcal{T}_A \rightarrow \mathcal{M}_A$, given by restricting homeomorphisms $(S^2, A) \rightarrow \widehat{\mathbb{C}}$ to A , which is a universal covering map with group $\mathbf{Mod}(S^2, A)$.

Thurston associates with $f: (S^2, C) \rightarrow (S^2, A)$ a map $\sigma_f: \mathcal{T}_A \rightarrow \mathcal{T}_C$ defined by pulling back complex structures through f , see [13]: given $h \in \mathcal{T}_A$ represented by a homeomorphism $h: (S^2, A) \rightarrow \widehat{\mathbb{C}}$, the composition $h \circ f$ defines a holomorphic atlas on $S^2 \setminus C$, with erasable singularities at C ; so there exists a homeomorphism $h': S^2 \rightarrow \widehat{\mathbb{C}}$ and a rational map f_h making the following diagram commutative:

$$(41) \quad \begin{array}{ccc} (S^2, C) & \xrightarrow{\quad h' \quad} & \widehat{\mathbb{C}} \\ \downarrow f & & \downarrow f_h \\ (S^2, A) & \xrightarrow{\quad h \quad} & \widehat{\mathbb{C}}. \end{array}$$

Furthermore, h' is uniquely defined up to post-composition by a Möbius transformation, and its class in \mathcal{T}_C depends only on the class of h in \mathcal{T}_A . The map σ_f is defined by $\sigma_f(h) = h'$. It is a contravariant functor from the category of spheres (S^2, A) with sphere maps as morphisms to the category of Teichmüller spaces \mathcal{T}_A with analytic maps as morphisms.

Unless f is a homeomorphism, the map $\sigma_f: \mathcal{T}_A \rightarrow \mathcal{T}_C$ does not descend to a map $\mathcal{M}_A \rightarrow \mathcal{M}_C$. However, define $H_f \leq \mathbf{Mod}(S^2, A)$, the subgroup of *liftable classes*, as

$$(42) \quad H_f = \{h \in \mathbf{Mod}(S^2, A) \mid \exists h' \in \mathbf{Mod}(S^2, C) \text{ with } h \circ f = f \circ h'\}.$$

By Proposition 6.8 or [22, Proposition 3.1], the subgroup H_f has finite index in $\mathbf{Mod}(S^2, A)$, and Proposition 6.4 implies that the h' in (42) is unique; see also [7]. Define then

$$\mathcal{W}_f = \mathcal{T}_A / H_f.$$

Then \mathcal{W}_f is a finite covering of \mathcal{M}_A , say with covering map i , and the map σ_f descends to a map $\overline{\sigma}_f: \mathcal{W}_f \rightarrow \mathcal{M}_C$. We therefore have a correspondence $(\overline{\sigma}_f, i)$, which we call *modular correspondence*, see [21],

$$(43) \quad \mathcal{M}_C \xleftarrow{\overline{\sigma}_f} \mathcal{W}_f \xrightarrow{i} \mathcal{M}_A$$

with the associated biset (see [3, §4.1 and Equation (15)])

$$(44) \quad B((\overline{\sigma}_f, i), \dagger, *) = \{(\delta: [0, 1] \rightarrow \mathcal{M}_C, z \in \mathcal{W}_f) \mid \delta(0) = \dagger, \delta(1) = \overline{\sigma}_f(z), i(z) = *\} / \approx.$$

Choose $\tilde{\dagger} \in \mathcal{T}_C$ and $\tilde{*} \in \mathcal{T}_A$ above $\dagger, *$ respectively; by definition they are maps $\tilde{\dagger}: (S^2, C) \rightarrow \widehat{\mathbb{C}}$ and $\tilde{*}: (S^2, A) \rightarrow \widehat{\mathbb{C}}$. Using these choices we have identifications $\pi_1(\mathcal{M}_C, \dagger) \cong \mathbf{Mod}(S^2, C)$ and $\pi_1(\mathcal{M}_A, *) \cong \mathbf{Mod}(S^2, A)$. Indeed, every

$\gamma \in \pi_1(\mathcal{M}_C, \dagger)$ determines an element of $\mathbf{Mod}(S^2, C)$ as the composition

$$(45) \quad C \hookrightarrow (S^2, C) \xrightarrow{\tilde{\dagger}} \dagger \xrightarrow{\tilde{\gamma}} \dagger \xrightarrow{\tilde{\dagger}^{-1}} (S^2, C)$$

with \dagger viewed as a marked Riemann sphere and $\dagger \xrightarrow{\tilde{\gamma}} \dagger$ viewed as an isotopy of marked Riemann spheres. Similarly, every $\gamma \in \pi_1(\mathcal{M}_A, *)$ determines an element of $\mathbf{Mod}(S^2, A)$ as the composition

$$(46) \quad A \hookrightarrow (S^2, A) \xrightarrow{\tilde{*}} * \xrightarrow{\tilde{\gamma}} * \xrightarrow{\tilde{*}^{-1}} (S^2, A).$$

The following result identifies biset elements $b = (\delta, z) \in B((\overline{\sigma}_f, i), \dagger, *)$ with branched coverings $b: (S^2, C) \rightarrow (S^2, A)$:

Proposition 8.1. *Suppose that $\tilde{\dagger} \in \mathcal{T}_C$ and $\tilde{*} \in \mathcal{T}_A$ are fixed as above and suppose that $\pi_1(\mathcal{M}_C, \dagger) \cong \mathbf{Mod}(S^2, C)$ and $\pi_1(\mathcal{M}_A, *) \cong \mathbf{Mod}(S^2, A)$ are identified by (45) and (46).*

*Then the biset of the modular correspondence (43) is isomorphic to the mapping class biset $M(f, C, A)$, see Definition 6.3, by an isomorphism mapping $(\delta, z) \in B((\overline{\sigma}_f, i), \dagger, *)$ given in the form of (44) to the following sphere map (47). Choose any $\tilde{z} \in \mathcal{T}_A$ above z . Then the sphere map*

$$(47) \quad b: (S^2, C) \xrightarrow{\tilde{\dagger}} \dagger \xrightarrow{\tilde{\delta}} \delta(1) \xrightarrow{\overline{\sigma}_f(\tilde{z})^{-1}} (S^2, C) \xrightarrow{f} (S^2, A) \xrightarrow{\tilde{z}} * \xrightarrow{\tilde{*}^{-1}} (S^2, A)$$

is independent up to isotopy of the choice of \tilde{z} . Here the map $\tilde{\delta}: [0, 1] \rightarrow \mathcal{T}_C$ is the lift of δ that ends at $\overline{\sigma}_f(\tilde{z})(C)$, and represents the continuous deformation of $(\hat{\mathbb{C}}, \tilde{\dagger}(C))$ into $(\hat{\mathbb{C}}, \overline{\sigma}_f(\tilde{z})(C))$.

Proof. Suppose that $\tilde{z}' \in \mathcal{T}_A$ is a different choice of a point above z . Then $\tilde{z}' = m\tilde{z}$ for some $m \in H_f$, see (42), and $fm = m'f$ for some $m' \in \mathbf{Mod}(S^2, C)$. By definition of σ_f , we have $\overline{\sigma}_f(m\tilde{z}) = m'\sigma(\tilde{z})$; therefore substitution of \tilde{z} by \tilde{z}' does not change the isotopy type of (42).

The map in question is a morphism of left-free transitive bisets; it is an isomorphism because the group of the bisets have the same group of liftable elements. \square

8.1. Twisted rabbits. We give now a few examples of calculations of the structure of mapping class bisets, following [1]. It was used in that article to solve “twisted rabbit” problems, of the following kind: consider the “rabbit” polynomial $f_R(z) = z^2 + c$ for $c \approx -0.12256 + 0.74486i$, with post-critical set $A = \{0, c, c^2 + c, \infty\}$. Let t denote a Dehn twist about a simple closed curve surrounding c and $c^2 + c$ (the “ears” of the rabbit). By Thurston’s Theorem 3.5 characterizing rational maps, the “twisted rabbit” $t^n \circ f_R$ is combinatorially equivalent to a degree-two polynomial, which may then be normalized in the form $z^2 + c$ with $(c^2 + c)^2 + c = 0$; so it is either f_R , $f_C := z^2 + \bar{c}$ (the “corabbit”), or $f_A \approx z^2 - 1.7549$ (the “airplane”). The problem is to determine which one.

The reformulation of this question is the following. The biset $M(f_R, A)$ is a left-free $\mathbf{Mod}(S^2, A)$ -biset of degree 2, and can be computed explicitly by Algorithm 6.11. By Thurston’s Theorem 3.5, there are precisely 3 conjugacy classes (in the sense of [3, §4.7]) in $M(f_R, A)$, containing respectively the rabbit, corabbit and airplane polynomials. As we shall see in [5], the conjugacy problem is solvable in $M(f_R, A)$.

The answer to the concrete question above is: to determine the conjugacy class of $t^n \circ f_R$, write n in base 4, as $n = \sum_{i \geq 0} n_i 4^i$ with $n_i \in \{0, 1, 2, 3\}$; if $n \geq 0$ then almost all $n_i = 0$ while if $n < 0$ then almost all $n_i = 3$. Then [1, Proposition 4.3]

$$(48) \quad t^n \circ f_R \sim \begin{cases} f_R & \text{if } n \geq 0 \text{ and all } n_i \in \{0, 3\}, \\ f_C & \text{if } n < 0 \text{ and all } n_i \in \{0, 3\}, \\ f_A & \text{if some } n_i \in \{1, 2\}. \end{cases}$$

The equalities in (48) follow from an explicit computation of the structure of $M(f_R, A)$. In this case, it is more direct to obtain it using Teichmüller theory, as it is done in [1, §5], than to run our general algorithm. Moduli space \mathcal{M}_A may be described as $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ by identifying $A = \{0, c, c^2 + c, \infty\}$ with its cross-ratio $c + 1$. The correspondence $\mathcal{M}_A \leftarrow \mathcal{W}_{f_R} \rightarrow \mathcal{M}_A$ from (43) descends in this case to the inverse of a rational map $g: \mathcal{M}_A \leftarrow \mathcal{M}_A$. Indeed, consider $h': (S^2, A) \rightarrow \widehat{\mathbb{C}}$ in Teichmüller space, and its image $\sigma_{f_R}(h') = h$. Let $\{0, 1, w, \infty\}$ be the image of h' in moduli space. Then in (41) the rational function f_h has degree 2, has critical points at $0, \infty$, and satisfies

$$f_h(\infty) = \infty, \quad f_h(0) = 1, \quad f_h(w) = 0$$

so $f_h(z) = 1 - z^2/w^2$; and the image $\{0, 1, w', \infty\}$ of h' in moduli space is given by $w' = f_h(1)$ so we have

$$g(w) = 1 - \frac{1}{w^2}.$$

The post-critical set of g is $\{0, 1, \infty\}$, and $M(S^2, A)$ is naturally identified with $\pi_1(\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}, w)$, by identifying the loop traced by w in $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ with the mapping class of $S^2 \setminus A$ in which the non-critical preimage of the critical point is dragged along the loop, keeping all other points in A fixed. We write $M(S^2, A) = \langle s, t, u \mid uts \rangle$, with s representing the positive Dehn twist about a curve surrounding $c^2 + c$ and 0 .

We have $M(f_R, A) \cong B(g)$, and may write its structure as follows, in basis $\{f_R, f_R t\}$, see Figure 2:

$$(49) \quad \begin{array}{lll} f_R \cdot t = f_R t, & f_R \cdot s = t \cdot f_R, & f_R \cdot u = s \cdot f_R t, \\ f_R t \cdot t = u \cdot f_R, & f_R t \cdot s = f_R t, & f_R t \cdot u = f_R. \end{array}$$

The claims in (48) then follow from the initial cases

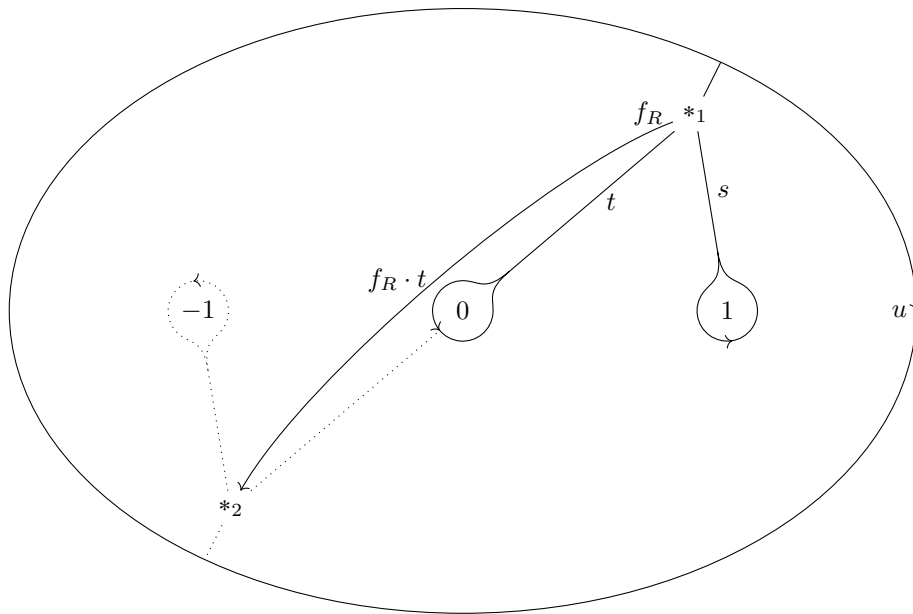
$$t^{-1} \cdot f_R \sim f_R t^{-1} \sim f_C, \quad t \cdot f_R \sim f_R t \sim f_A$$

which can be checked by hand, and the equations

$$\begin{aligned} t^{4k} f_R \sim f_R t^{4k} &= u^{2k} f_R \sim f_R u^{2k} = s^k f_R \sim f_R s^k &= t^k f_R, \\ t^{4k+1} f_R \sim f_R t^{4k+1} &= u^{2k} f_R t \sim f_R t u^{2k} = s^k f_R t \sim f_R t s^k &= f_R t, \\ t^{4k+2} f_R \sim f_R t^{4k+2} &= u^{2k+1} f_R \sim f_R u^{2k+1} = s^{k+1} f_R t \sim f_R t s^{k+1} &= f_R t, \\ t^{4k+3} f_R \sim f_R t^{4k+3} &= u^{2k+1} f_R t \sim f_R t u^{2k+1} = s^k f_R \sim f_R s^k &= t^k f_R \end{aligned}$$

which directly follow from (49).

Similar calculations were performed in [1, §6] to study $M(z^2 + i, A, A)$ with $A = \{i, i - 1, -i, \infty\}$. Again this biset is left-free of degree 2; and coincides with the biset of the rational map $g(w) = (1 - 2/w)^2$ after \mathcal{M}_A has been identified with $\widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$. There are infinitely many conjugacy classes in $M(z^2 + i, A, A)$,



see [1, Corollary 6.11]: two classes corresponding to rational maps $z^2 + i$ and $z^2 - i$, and a \mathbb{Z} 's worth of conjugacy classes corresponding to obstructed maps f_n with $n \in \mathbb{Z}$, as described in [2, §7.2].

In [4] we shall consider the more general situation of a map $f: (S^2, C) \rightarrow (S^2, A)$ and its restriction $f': (S^2, C') \rightarrow (S^2, A')$ for $C' \subset C, A' \subset A$; the philosophy being that $M(f, C, A)$ is an extension of $M(f', C', A')$ via the natural maps $\mathbf{Mod}(S^2, C) \twoheadrightarrow \mathbf{Mod}(S^2, C')$, $\mathbf{Mod}(S^2, A) \twoheadrightarrow \mathbf{Mod}(S^2, A')$ and $M(f, C, A) \twoheadrightarrow M(f', C', A')$, using which $M(f, C, A)$ can be efficiently encoded using $M(f', C', A')$ and $B(f')$. In our case $M(f', C', A')$ is trivial and the considerations are much simplified.

Let us fix a basepoint $* \in S^2 \setminus A$, and write $\pi = \pi_1(S^2 \setminus A, *)$. It admits a presentation as a sphere group

generated by loops around the punctures $0, 1, \infty, v$ respectively. Then the covering $f: S^2 \setminus f^{-1}(A) \rightarrow S^2 \setminus A$ is described by an index-deg(f) subgroup $\pi_f \leq \pi$: fix

a preimage $*_0$ of $*$, and let π_f denote those loops in π that lift to a loop at $*_0$. Equivalently, $S^2 \setminus A \cong \pi \setminus \mathbb{D}$ and $S^2 \setminus f^{-1}(A) \cong \pi_f \setminus \mathbb{D}$ with $f: \pi_f \setminus \mathbb{D} \rightarrow \pi \setminus \mathbb{D}$ the natural map.

There is a natural map $\pi = \pi_1(S^2 \setminus A, *) \rightarrow \pi_1(S^2 \setminus \{0, 1, \infty\}, *) =: G$; let H denote the image of π_f in G . We have $G = \pi / \langle d \rangle^\pi$; and $\pi_f \geq \langle d \rangle^\pi$ since d lifts to d loops at every preimage of $*$; so $[G : H] = [\pi : \pi_f] = \deg(f)$, and we may also represent f' as the quotient map $H \setminus \mathbb{D} \rightarrow G \setminus \mathbb{D}$.

By choosing a path from $*$ to v , we may also identify G with $\pi_1(S^2 \setminus \{0, 1, \infty\}, v)$. Let $\{c_1, \dots, c_d\} \subset f^{-1}(A)$ be the preimages of v . Then G acts by monodromy on $\{c_1, \dots, c_d\}$, so that we have a transitive permutation representation $\rho: G \rightarrow d!$. A choice of $c_1 \in f^{-1}(A)$ determines a choice of H as the stabilizer of c_1 under ρ . The pure modular group $\mathbf{Mod}(S^2, A)$ is free of rank 2, and is identified with G : the element of G represented by a simple loop of v turning respectively around $0, 1, \infty$ corresponds to the Dehn twist in $\mathbf{Mod}(S^2, A)$ in which v is dragged respectively around $0, 1, \infty$.

We may also present G as a sphere group,

$$(50) \quad G = \mathbf{Mod}(S^2, A) = \langle s, t, u \mid uts \rangle.$$

It is the group of outer automorphisms of π that fixes conjugacy classes of generators. The outer automorphisms s, t, u are, modulo interior automorphisms,

$$\begin{aligned} s: & a \mapsto a^{ad}, & b & \mapsto b, & c & \mapsto c, & d & \mapsto d^{ad}, \\ t: & a \mapsto a, & b & \mapsto b^{db}, & c & \mapsto c^{[d,b]}, & d & \mapsto d^{db}, \\ u: & a \mapsto a, & b & \mapsto b, & c & \mapsto c^{dc}, & d & \mapsto d^{dc}. \end{aligned}$$

These generators are respectively Dehn twists about the curves ad, db, dc .

Lemma 8.2. *The group of liftable elements $H_f \leq G$ is the kernel of $\rho: G \rightarrow d!$.*

More precisely, denote by $\mathbf{Mod}^(S^2, C)$ the group of not necessarily pure mapping classes of (S^2, C) that fix $0, 1, \infty$. Then $\mathbf{Mod}^*(S^2, C)$ extends the left action of $\mathbf{Mod}(S^2, C)$ on $M(f, C, A)$ and the new extended action is free and transitive. For every $h \in G$ there is a unique $\tilde{h} \in \mathbf{Mod}^*(S^2, C)$ such that $\tilde{h}f = fh$. The element h is liftable if and only if $\tilde{h} \in \mathbf{Mod}(S^2, C)$.*

Proof. We show that the action of $\mathbf{Mod}^*(S^2, C)$ on $M(f, C, A)$ is free; the other claims then follow easily. Suppose that $hf = f$ in $M(f, C, A)$ for $h \in \mathbf{Mod}^*(S^2, C)$. There is an $e \geq 1$ such that $h^e \in \mathbf{Mod}(S^2, C)$. We have $h^e f = f$, so by Proposition 6.4 we have $h^e = 1$, so h is isotopic to a Möbius transformation. Since h fixes $0, 1, \infty$ we get $h = 1$ in $\mathbf{Mod}^*(S^2, C)$. \square

Fix f' -preimages $0', 1', \infty'$ of $0, 1, \infty$ respectively, and write $\mathcal{T}' = \mathcal{T}_{\{0', 1', \infty', c_1, \dots, c_d\}}$ and $\mathcal{M}' = \mathcal{M}_{\{0', 1', \infty', c_1, \dots, c_d\}}$. Once $0, 1, \infty$ and the points above them are fixed, the map f_h in (41) is uniquely determined, so we may consider $\mathcal{T}', \mathcal{M}'$ rather than the larger $\mathcal{T}_{f^{-1}(A)}, \mathcal{M}_{f^{-1}(A)}$. In the following diagram, vertical arrows are coverings:

$$(51) \quad \begin{array}{ccccc} \mathcal{T}_C & \longleftarrow & \mathcal{T}' & \longleftarrow & \mathcal{T}_A \\ & & \downarrow & & \downarrow \\ & & \mathcal{M}' & \longleftrightarrow & \mathcal{W}_f \\ & \swarrow & & \searrow & \\ \mathcal{M}_C & & & & \mathcal{M}_A \end{array}$$

The space \mathcal{M}' is naturally a subset of affine space:

$$(52) \quad \mathcal{M}' = \{(c_1, \dots, c_d) \in \widehat{\mathbb{C}}^d \mid c_i \neq 0', 1', \infty'; c_i \neq c_j\},$$

and the embedding of \mathcal{W}_f in \mathcal{M}' is onto the curve

$$\mathcal{W}_f = \{(c_1, \dots, c_d) \in \widehat{\mathbb{C}}^d \mid f(c_1) = \dots = f(c_d) \neq 0, 1, \infty\}.$$

The correspondence is then determined by the map $\mathcal{M}' \rightarrow \mathcal{M}_C$, which encodes how $f^{-1}(A)$ corresponds to C .

8.2.1. Products of mapping class bisets. Write $A = \{0, 1, \infty, v\}$, consider the map $f(z) = z^5: (\widehat{\mathbb{C}}, \{0, 1, \infty, f^{-1}(v)\}) \rightarrow (\widehat{\mathbb{C}}, A)$, set $C = \{0, 1, \infty, f^{-1}(v)\} = f^{-1}(A)$ and denote by $\{c_1, \dots, c_5\}$ the five f -preimages of v . Choose a homeomorphism $i: (\widehat{\mathbb{C}}, A) \rightarrow (\widehat{\mathbb{C}}, \{c_1, c_2, c_3, c_4\})$. Denote also by $k = \mathbb{1}: (\widehat{\mathbb{C}}, \{c_1, c_2, c_3, c_4\}) \rightarrow (\widehat{\mathbb{C}}, C)$ the map marking $0, 1, \infty, c_5$, and set $j = ik$. Denote also $\mathbf{Mod}(\widehat{\mathbb{C}}, A)$ by G . We shall compute $M(f, C, A)$, $M(j, A, C)$ and $M(jf, A, A)$ and note that the natural map $M(j, A, C) \otimes M(f, C, A) \rightarrow M(jf, A, A)$ is not an isomorphism of G - G -bisets.

Since f is a Belyi map, we determine $M(f, C, A)$ using Lemma 8.2. Since $\rho(G)$ is cyclic of order 5, the biset $M(f, C, A)$ is left-free of degree 5. Also, every mapping class $n \in G$ is liftable by f , but possibly to an impure class: there exists $n' \in \mathbf{Mod}^*(\widehat{\mathbb{C}}, \{0, 1, \infty, f^{-1}(v)\})$ with $n'f = fn$, and n' acts on $\{c_1, \dots, c_5\}$ by a power of the cycle $(1, 2, 3, 4, 5)$.

The maps i and k are homeomorphisms, so $M(j, A, C)$ is the dual of the forgetful homomorphism j^* induced on mapping class groups, see Lemma 6.10. In particular, $M(j, A, C)$ is left-principal. The product $M(j, A, C) \otimes M(f, C, A)$ is therefore left-free of degree 5.

We claim that the biset $M(jf, A, A)$ is left-principal. Indeed, we may write $jf = g\ell$ with $g = z^5 \circ i: (\widehat{\mathbb{C}}, A) \rightarrow (\widehat{\mathbb{C}}, \{0, \infty, v\})$ and $\ell = \mathbb{1}: (\widehat{\mathbb{C}}, \{0, \infty, v\}) \rightarrow (\widehat{\mathbb{C}}, A)$. Consider $n \in G$; then we have $jfn = g\ell n = g\ell = jf$, since $\mathbf{Mod}(\widehat{\mathbb{C}}, \{0, \infty, v\})$ is trivial and ℓ^{-1} is an erasing map.

We also remark that the pullback map $\sigma_{jf} = \sigma_{g\ell}$ is constant, because σ_ℓ is constant; so the modular correspondence \mathcal{W}_{jf} coincides with \mathcal{M}_A , with induced map $\overline{\sigma_{jf}}$ constant. Its image point is the cross-ratio of four fifth roots of unity. The diagram (51) becomes

$$\begin{array}{ccccc} \mathcal{T}_A & \xleftarrow{\sigma_j} & \mathcal{T}' & \xleftarrow{\sigma_f} & \mathcal{T}_A \\ \downarrow & & \downarrow & & \downarrow \\ & & \mathcal{M}' & \longleftrightarrow & \mathcal{W}_f \\ & \swarrow & & \searrow & \\ \mathcal{M}_A & \xleftarrow{\overline{\sigma_{jf}}} & \mathcal{W}_{jf} & \xlongequal{5:1} & \mathcal{M}_A \end{array}$$

8.2.2. Kevin Pilgrim's "blowing up an arc" map. We consider another concrete example illustrating §8.2: a degree-5 self-map g , due to Kevin Pilgrim, which is obtained from a torus endomorphism by blowing up an arc; see [2, §7.6], repeated for convenience in Figure 3 as the correspondence (i, f) . Note that we exchanged b and c compared to [2, Figure 5].

The map g is non-dynamically modeled on the map $f(z) = z^3((4z + 5)/(5z + 4))^2$ viewed as a Belyi map $(\widehat{\mathbb{C}}, C) \rightarrow (\widehat{\mathbb{C}}, A)$ with $A = \{0, 1, \infty, v\}$ and $C =$

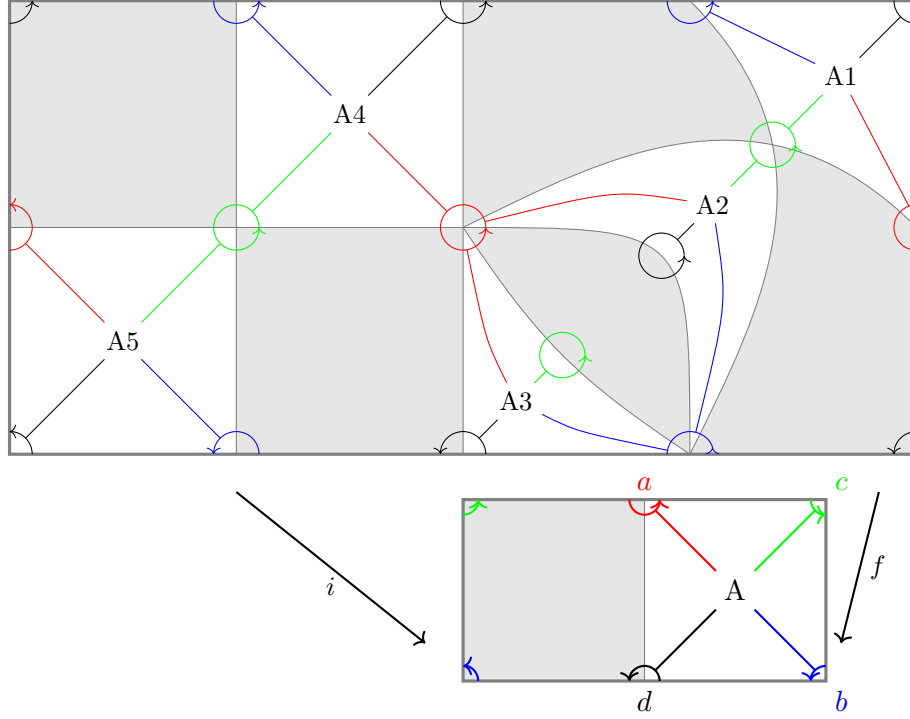


FIGURE 3. Pilgrim's “blowing up an arc” subdivision rule

$\{0, 1, \infty, f^{-1}(v)\}$; with such choices the map \mathcal{M}' and \mathcal{M}_C are identical via the map in Diagram (51). Recall that Pilgrim's map is the map $g: (S^2, A) \hookrightarrow$ obtained (topologically) from $f: (\widehat{\mathbb{C}}, C) \rightarrow (\widehat{\mathbb{C}}, A)$ by identifying A with four preimages c_1, \dots, c_4 of v . We thus have a map $\sigma': \mathcal{M}_C \approx \mathcal{M}' \rightarrow \mathcal{M}_A$ given by evaluating the cross-ratio of the first four coordinates of $\{c_1, \dots, c_5\} \in \mathcal{M}'$, see Diagram (54).

Let $\mathcal{M}_A \xleftarrow{\overline{\sigma}_g} \mathcal{W}_f \xrightarrow{i} \mathcal{M}_A$ be the correspondence obtained from $\mathcal{M}_C \leftarrow \mathcal{W}_f \rightarrow \mathcal{M}_A$ in Diagram (51) by precomposing with $\mathcal{M}_A \xleftarrow{\sigma'} \mathcal{M}_C$. The correspondence we obtain is explicitly given by the pair of maps

$$(53) \quad \begin{aligned} i(c_1, \dots, c_5) &= f(c_1), \\ \sigma'(c_1, \dots, c_5) &= \frac{(c_1 - c_3)(c_2 - c_4)}{(c_1 - c_4)(c_2 - c_3)} =: x; \end{aligned}$$

We shall see that it is the modular correspondence of g . On the other hand, we shall also see that \mathcal{W}_g is not the minimal correspondence above \mathcal{M}_A covered by the

map σ_g . Thus diagram (51) will become

$$(54) \quad \begin{array}{ccccc} \mathcal{T}_A & \longleftarrow & \mathcal{T}_C & \longleftarrow & \mathcal{T}_A \\ & & \downarrow & & \swarrow \\ & & \mathcal{M}_C & \longleftrightarrow & \mathcal{W}_f = \mathcal{W}_g \\ & \swarrow \sigma' & & \searrow & \downarrow \\ \mathcal{M}_A & & & & \mathcal{W} \\ & \nwarrow \overline{\sigma_g} & & \nearrow & \searrow \\ & & & & \mathcal{M}_A \end{array} \quad \begin{array}{l} \sigma_f \\ 5! : 1 \\ 30 : 1 \end{array}$$

Let us compute the biset of the map g . With $\pi = \pi_1(\widehat{\mathbb{C}} \setminus A)$, it is the following π - π -biset $B(g)$, in the basis $X = \{1, 2, 3, 4, 5\}$:

$$(55) \quad \begin{aligned} a &\mapsto \ll a, 1, 1, 1, a^{-1} \gg (1, 3, 5)(2, 4) \\ b &\mapsto \ll b^{-1}, 1, 1, b, 1 \gg (1, 4)(2, 5, 3) \\ c &\mapsto \ll 1, 1, c, c^{-1}, 1 \gg (1, 2)(3, 4) \\ d &\mapsto \ll a, b, d, c, 1 \gg. \end{aligned}$$

Let us write $G = \mathbf{Mod}(\widehat{\mathbb{C}}, A)$. To compute the monodromy representation ρ of f , we simply consider the permutations in the biset $B(g)$. We present G as a sphere group $\langle s, t, u \mid uts \rangle$; the twists s, t, u correspond to a motion of v around $0, \infty, 1$ respectively. Therefore, $\rho: G \rightarrow 5\downarrow$ is given by

$$s \mapsto (1, 3, 5)(2, 4), \quad t \mapsto (1, 4)(2, 5, 3), \quad u \mapsto (1, 2)(3, 4).$$

In particular, ρ is onto $5\downarrow$ so the subgroup of liftables $H_f = \ker(\rho)$ from (50) has index 120, and both the mapping class biset $M(f, C, A)$ and the modular correspondence \mathcal{W}_f have degree 120.

Let us consider the G - G -biset $M(g, A) := \{B(\phi g \psi)\} / \sim$ of isomorphism classes of twists of the biset $B(g)$. This biset is left-free, and a basis may be found by considering its distillations, see Definition 6.6. Since $\rho(G) = 5\downarrow$, its centralizer in $5\downarrow$ is trivial, so the permutations associated with the generators a, b, c in the biset $B(\phi g \psi)$ can in a unique manner be brought respectively to $(1, 3, 5)(2, 4)$, $(1, 4)(2, 5, 3)$, $(1, 2)(3, 4)$. The conjugacy classes on the five entries of d in $B(\phi g \psi)$ are then $1, a, b, c, d$ in some order, and this order determines uniquely an element of $5\downarrow$. All such orderings may appear, so $M(g, A)$ is also left-free of degree 120.

Using a choice of distillations, Algorithm 6.11 lets us compute a presentation for $M(g, A)$, and in this manner describe the correspondence. We shall not describe $M(g, A)$ in such a basis, but will content ourselves with giving, for illustration, the results of the calculation in the form of the lifts of the conjugacy classes s^G, t^G, u^G .

Since \mathcal{M}_A is a sphere punctured at $\{0, 1, \infty\}$, the modular correspondence is a sphere correspondence. Recall from §2.1.1 that to each of the peripheral conjugacy classes s^G, t^G, u^G can be assigned the multiset $\{(d_i, h_i^G) \mid i = 1, \dots, \ell\}$ of its lifts.

Let us start with u^G . The degrees d_i are all the same, and equal to the order of $\rho(u)$, so there are 60 cycles of length 2, given by the regular action of $\rho(u)$ on $5\downarrow$. The lifts are $16 \times s^G$, $16 \times t^G$, $16 \times u^G$, $4 \times (s^2)^G$, $4 \times (t^2)^G$, $4 \times (u^2)^G$. Similarly, the degrees of the lifts of s^G and t^G are all 6 since $\rho(s)$ and $\rho(t)$ have order 6, and both s^G and t^G lift to 8×1^G , $4 \times (s^5)^G$, $4 \times (t^5)^G$ and $4 \times (u^5)^G$. We deduce that the

degree of the map $\overline{\sigma_g}: \mathcal{W}_f \rightarrow \mathcal{M}_A$ is $16 \times 1 + 4 \times 2 + 4 \times 5 + 4 \times 5 = 64$, counting the number with degree and multiplicity of $(s^n)^G$ that appear as lifts of s^G, t^G, u^G . In particular, the correspondence is not constant, and moreover is given by a pair of branched coverings $\mathcal{W}_f \rightrightarrows \mathcal{M}_A$.

In this example, there exists a correspondence $\mathcal{M}_A \leftarrow \mathcal{W} \rightarrow \mathcal{M}_A$ which is a quotient of $\mathcal{M}_A \leftarrow \mathcal{W}_f \rightarrow \mathcal{M}_A$, and which also covers $\sigma_g: \mathcal{T}_A \hookrightarrow$. In fact, we shall give the minimal such \mathcal{W} , and show that it has left-degree 30.

The variables c_2, \dots, c_5 may be eliminated from \mathcal{W}_f , yielding a planar projection of the correspondence. It is given by a polynomial equation $P(v, x) = 0$ of degree 30 in x and 16 in v , see (53); so that the degree of the map towards $v = f(c_1)$ is 30, and the degree of the map towards $x = [c_1, c_2; c_3, c_4]$ is 16. The polynomial $P(v, x)$ may be computed using Gröbner bases as follows, in MAPLE using the package FGB¹:

```
with(FGb);
sys := [(c[1]-c[3])*(c[2]-c[4])-x*(c[1]-c[4])*(c[2]-c[3]),
  coeff(16*product(z-c[i], i=1..5)-z^3*(4*z+5)^2+v*(5*z+4)^2, z, i)$i=0..4]:
fgb_gbasis_elim(sys, 0, [c[i]$i=1..5], [x, v], {"verb"=3, "index"=2000000}):
P := simplify(%[1]/v):
lprint(P);
```

One can easily check that P is irreducible, so $\mathcal{M}_A \leftarrow \mathcal{W} \rightarrow \mathcal{M}_A$ is indeed the minimal correspondence covered by σ_g .

Here is the explanation of the existence of a smaller cover \mathcal{W} , using group theory. The minimality of \mathcal{W} follows from the fact that the induced map on conjugacy classes admits an order-4 symmetry and no larger one. The cross-ratio map σ' from (53) admits symmetries: $\sigma'(c_1, c_2, c_3, c_4, c_5) = \sigma'(c_2, c_1, c_4, c_3, c_5) = \sigma'(c_3, c_4, c_1, c_2, c_5) = \sigma'(c_4, c_3, c_2, c_1, c_5)$. We consider

$$V = \{1, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$$

the Klein group of order 4, so that σ' is invariant under permutation of its arguments by V . We define $\mathcal{W} = \mathcal{T}_A / \rho^{-1}(V)$; then the map $\overline{\sigma_g}: \mathcal{W}_g \rightarrow \mathcal{M}_A$ descends to a map $\mathcal{W} \rightarrow \mathcal{M}_A$. Thus V acts on \mathcal{W}_g , and the quotient is \mathcal{W} .

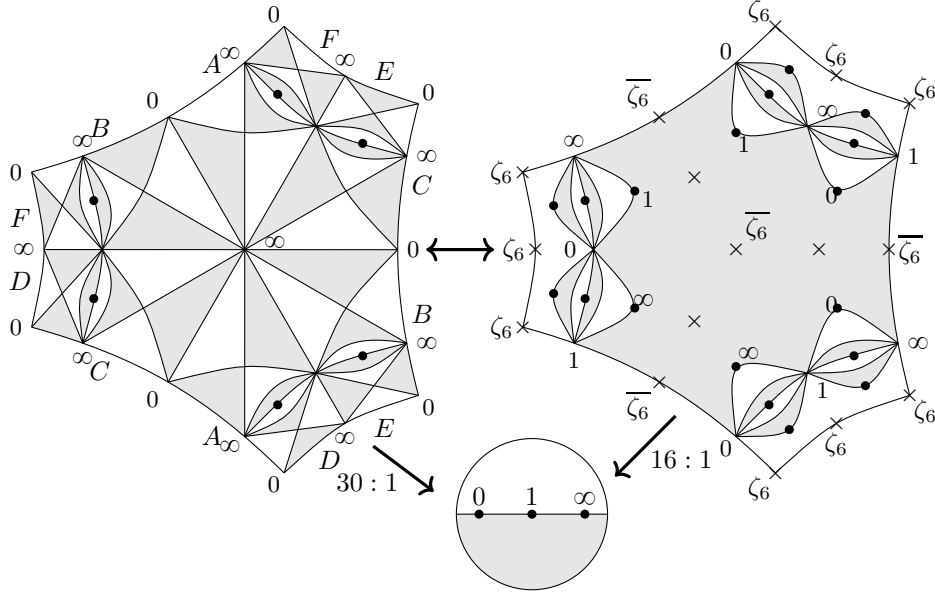
The Klein group V acts regularly on the set of distillations of bisets in $M(g, A)$; this action combines with the left action of G on $M(g, A)$ to give a left action of $V \times G$ on $M(g, A)$, by impure mapping classes, which is still free, but now of degree 30. The quotient biset $V \backslash M(g, A)$ is a left-free G - G -biset of degree 30, and coincides with the biset $B(\mathcal{M}_A \leftarrow \mathcal{W} \rightarrow \mathcal{M}_A)$ of the quotient correspondence.

Recall that the map $g: (S^2, A) \hookrightarrow$ was obtained from the doubling map $z \mapsto 2z$ on the torus $\mathbb{C}/\mathbb{Z} + \mathbb{Z}i$, by blowing up an arc. The Klein group V acts on the torus by $\langle z \mapsto z + \frac{1}{2}, z \mapsto z + \frac{i}{2} \rangle$, and elements of the biset $V \backslash M(g, A)$ can be identified with orbits of V under this action.

We repeat the analysis of the action of peripheral elements on $M(g, A)/V$, and derive information on \mathcal{W} . It is readily checked that the elements s, t act on the coset space $5\mathbb{A}/V$ as a product of five 6-cycles, each carrying the conjugacy classes $(s^5)^G, (t^5)^G, (u^5)^G, 1^G, 1^G$; and that u acts as a product of twelve involutions (with six fixed points).

Therefore, the correspondence \mathcal{W} has five degree-6 punctures above 0, mapping respectively by degree 5 to 0, 1, ∞ and two other points (the sixth roots of unity); it

¹available from <http://www-polsys.lip6.fr/~jcf/FGb/>


 FIGURE 4. The modular correspondence of g .

has similarly five degree-6 punctures above ∞ , mapping respectively by degree 5 to $0, 1, \infty$ and the sixth roots of unity; and eighteen punctures above 1, twelve of degree 2 and six of degree 1, mapping by degree 1 to $0, 1, \infty$ six times each. The surface \mathcal{W} has therefore 28 punctures and its Euler characteristic is $30 \cdot (2 - 3) = -30$, so \mathcal{W} has genus 2. The correspondence is given in Figure 4, in the standard description of holomorphic maps by shading the lower half plane.

The central component of the figure on the right is homeomorphic to a punctured torus; it maps $4 : 1$ to the disk, with the boundary mapping $4 : 1$, one point (the sixth root of unity ζ_6) having two order-2 preimages, and three points having one order-2 preimage (marked by a simple \times) and two regular preimages.

8.3. A Thurston map with infinitely generated centralizer. Our last example shows that centralizers of Thurston maps can be sometimes quite complicated, and in particular not finitely generated (whence our notion of “sub-computable”). We will also explicitly compute a biset of the form $M(f, A, \mathcal{C})$.

8.3.1. General construction. We consider a Thurston map $f : (S^2, A, \mathcal{C}) \hookrightarrow$ with $A = A_0 \sqcup A_1 \sqcup A_2$. The map admits an annular obstruction $\mathcal{C} = \{s, t\}$, with s separating A_0 from $A_1 \cup A_2$ and t separating A_2 from $A_0 \cup A_1$. The small spheres S_0, S_1, S_2 containing A_0, A_1, A_2 respectively are fixed by f , which acts on S_0 and S_2 as the identity and acts on S_1 as a rational map of degree 2.

All f -preimages of s and t map by degree 1. The curve s has a unique essential preimage that is isotopic to s while t has $2\ell \geq 2$ preimages isotopic to s and $2\ell + 1 \geq 3$ preimages isotopic to t . The Thurston matrix of f is therefore

$$T_{f, \mathcal{C}} = \begin{pmatrix} 1 & 2\ell \\ 0 & 2\ell + 1 \end{pmatrix}.$$

The case $\ell = 1$ is show in Figure 5.

We assume that all trivial f -preimages of S_0, S_1, S_2 are of degree 1 and all annular f -preimages of S_1, S_2 are of degree 2. (We remark that S_0 has no annular preimages.) The degree of f is thus $4\ell + 2$.

The spheres S_1 has ℓ annular preimages isotopic to s and ℓ annular preimages isotopic to t . We denote them by $U_1, \dots, U_{2\ell}$ in order of increasing distance to S_0 . Similarly, S_2 has 2ℓ annular preimages isotopic to s or t , denoted $T_1, \dots, T_{2\ell}$ in order of increasing distance to S_0 .

Let us next assume that $\#A_1 = 2$ and that A_1 is fixed by f . Therefore, we may normalize the restriction of f to S_1 as $z^2: (\hat{\mathbb{C}}, \{0, \infty, 1, -1\}) \hookrightarrow$ so that 1 and -1 correspond to curves t and s respectively. For $i > 1$ we normalize $f: \hat{U}_i \rightarrow \hat{S}_1$ as $z^2: (\hat{\mathbb{C}}, \{1, -1\}) \rightarrow (\hat{\mathbb{C}}, \{0, \infty, 1, -1\})$ so that $1, -1 \in \hat{U}_i$ encode curves isotopic to s if $i \leq \ell$, and isotopic to t if $i > \ell$. Finally we normalize $f: \hat{U}_1 \rightarrow \hat{S}_1$ as $z^2: (\hat{\mathbb{C}}, \{1, i\}) \rightarrow (\hat{\mathbb{C}}, \{0, \infty, 1, -1\})$ so that $1, i \in \hat{U}_1$ encode curves isotopic to s .

Recall that the restrictions of f to S_0 and S_2 are set to be the identity. We assume $\#A_2 \geq 3$ and we mark the points $\infty, x_6, x_7 \in A_2$. For $i \leq \ell$ we normalize $f: \hat{T}_i \rightarrow \hat{S}_2$ as $\frac{z^2+x_6}{1+x_6}: (\hat{\mathbb{C}}, \{1, -1\}) \rightarrow (\hat{\mathbb{C}}, A_2 \cup \{1\})$ so that $x_6, \infty \in A_2$ are the critical values. The points $1, -1 \in \hat{T}_i$ encode curves isotopic to s while $1 \in \hat{S}_2$ encodes the curve t . For $i > \ell$ we normalize $f: \hat{T}_i \rightarrow \hat{S}_2$ as $\frac{z^2+x_7}{1+x_7}: (\hat{\mathbb{C}}, \{1, -1\}) \rightarrow (\hat{\mathbb{C}}, A_2 \cup \{1\})$ so that $x_7, \infty \in A_2$ are the critical values. The points $1, -1 \in \hat{T}_i$ encodes curves isotopic to t and $1 \in \hat{S}_2$ encodes the curve t .

8.3.2. The mapping class biset $M(f, A, \mathcal{C})$. We describe $M(f, A, \mathcal{C})$ following the recipe of §6.5. Write $\mathbf{Mod}(S^2, A, \mathcal{C}) = \mathbf{eMod}(S^2, A, \mathcal{C}) \times \mathbf{vMod}(S^2, A, \mathcal{C})$ with $\mathbf{eMod}(S^2, A, \mathcal{C}) \cong \mathbb{Z}^{\mathcal{C}}$ and $\mathbf{vMod}(S^2, A, \mathcal{C}) \cong \mathbf{Mod}(\hat{S}_0) \times \mathbf{Mod}(\hat{S}_1) \times \mathbf{Mod}(\hat{S}_2)$.

Let us first compute $G_f \leq \mathbf{vMod}(S^2, A, \mathcal{C})$, see (28). Write $G_f = G_{f,0} \times G_{f,1} \times G_{f,2}$ with $G_{f,i} \leq \mathbf{Mod}(\hat{S}_i)$. Since $f: (\hat{S}_1, \{0, \infty, 1, -1\}) \hookrightarrow$ factors as $(\hat{S}_1, \{0, \infty, 1, -1\}) \xrightarrow{z^2} (\hat{S}_1, \{0, \infty, 1\}) \xrightarrow{1} (\hat{S}_1, \{0, \infty, 1, -1\})$, every element in $\mathbf{Mod}(\hat{S}_1)$ is liftable through $f: (S_1, \{0, \infty, 1, -1\}) \hookrightarrow$; the lift is trivial in $\mathbf{Mod}(\hat{S}_1)$ but needs not be trivial in $\mathbf{Mod}(S^2, A, \mathcal{C})$. Namely, a Dehn twist about a curve surrounding $1, -1$ lifts to a Dehn twist around 1 which encodes t . By the same reasoning, every element in $\mathbf{Mod}(\hat{S}_1)$ is liftable trough $f: (\hat{U}_i, \{1, -1\}) \rightarrow (\hat{S}_1, \{0, \infty, 1, -1\})$ for $i > 1$. On the other hand, the subgroup of liftable elements in $\mathbf{Mod}(\hat{S}_1)$ under $f: (\hat{U}_1, \{1, i\}) \rightarrow (\hat{S}_1, \{0, \infty, 1, -1\})$ has index two in $\mathbf{Mod}(\hat{S}_1)$: a Dehn twist about a simple essential closed curve in $(\hat{S}_2, \{0, \infty, 1, -1\})$ separating 1 from -1 lifts to a half twist about the simple closed curve in $(\hat{U}_1, \{1, i\})$ separating 1 from i ; this gives an index two condition. Therefore, $G_{f,1} \leq \mathbf{Mod}(\hat{S}_1)$ has index two.

Since $f: (\hat{T}_i, \{1, -1\}) \rightarrow (\hat{S}_2, A_2 \cup \{1\})$ factors as $(\hat{T}_i, \{0, \infty, 1, -1\}) \xrightarrow{f} (\hat{S}_2, \{f(0), \infty, 1\}) \xrightarrow{1} (\hat{S}_2, A_2 \cup \{1\})$, every element in $\mathbf{Mod}(\hat{S}_2)$ is liftable trough $f: (\hat{T}_i, \{1, -1\}) \rightarrow (\hat{S}_2, A_2 \cup \{1\})$. Since the restrictions of f to S_0 and to S_2 are the identity, every element in $\mathbf{Mod}(\hat{S}_0)$ and in $\mathbf{Mod}(\hat{S}_2)$ is liftable through the global map $f: (S^2, A, \mathcal{C}) \hookrightarrow$. Therefore, $G_{f,0} = \mathbf{Mod}(\hat{S}_0)$ and $G_{f,2} = \mathbf{Mod}(\hat{S}_2)$.

Since all f -preimages of s and t map by degree 1, the group Λ from (27) is $\mathbb{Z}^{\mathcal{C}}$ with the actions of $\mathbb{Z}^{\mathcal{C}}$ given by $n_1 \cdot b \cdot n_2 = n_1 + b + T_{f,\mathcal{C}}(n_2)$. The map

$\theta_f: G_{f,0} \times G_{f,1} \times G_{f,2} \rightarrow \Lambda \cong \mathbb{Z}^{\mathcal{C}}$ is computed as
(56)

$$\theta_f(n_0, n_1, n_2) = (\theta'_1(n_1) + (2\ell - 1)\theta_1(n_1) + 2\ell\theta_{2,s}(n_2), (2\ell + 1)\theta_1(n_1) + 2\ell\theta_{2,t}(n_2)) \in \mathbb{Z}^{\{s,t\}}$$

with $\theta'_1, \theta_1, \theta_{2,s}, \theta_{2,t}$ defined as follows.

The map $\theta'_1: G_{f,1} \rightarrow \mathbb{Z}$ corresponds to lifting through $f: (\hat{U}_1, \{1, i\}) \rightarrow (\hat{S}_1, \{0, \infty, 1, -1\})$. If n be a positive Dehn twist about an essential simple closed curve c in $(\hat{S}_2, \{0, \infty, 1, -1\})$, then $\theta'_1(n) = 0$ if c does not separate 1 from -1 , and $\theta'_1(n^2) = 1$ otherwise. (Recall that in the last case $n \notin G_{f,1}$).

The map $\theta_1: G_{f,1} \rightarrow \mathbb{Z}$ is the restriction of $\theta_1: \mathbf{Mod}(\hat{S}_1) \rightarrow \mathbb{Z}$ defined in the following way. If n be a Dehn twist about an essential simple closed curve c in $(\hat{S}_1, \{0, \infty, 1, -1\})$, then $\theta_1(n) = 1$ if c is a peripheral curve around 1 in $(\hat{S}_1, \{0, \infty, 1\})$, and $\theta_1(n) = 0$ otherwise. In the former case c has exactly $2\ell - 1$ lifts isotopic to s (not counting lifting through $f: (\hat{U}_1, \{1, i\}) \rightarrow (\hat{S}_1, \{0, \infty, 1, -1\})$) and $2\ell + 1$ lifts isotopic to t . In the latter case none of the lifts of c is isotopic to a curve in $\{s, t\}$.

Suppose n is a Dehn twist about a curve c in $(\hat{S}_2, A_2 \cup \{1\})$. If c be a peripheral curve in $(\hat{S}_2, \{x_6, \infty, 1\})$ around 1, then $\theta_{2,s}(n) = 1$; otherwise $\theta_{2,s}(n) = 0$. In the former case c has exactly 2ℓ lifts isotopic to s . Similarly, if c is a peripheral curve in $(\hat{S}_2, \{x_7, \infty, 1\})$ around 1, then $\theta_{2,t}(n) = 1$; otherwise $\theta_{2,t}(n) = 0$. In the former case c has exactly 2ℓ lifts isotopic to t .

Denote by $\mathbf{Mod}(\hat{S}_1)M'_{G_{f,1}}$ the mapping class biset of $f: \hat{S}_1 \hookrightarrow$ with the right action restricted to $G_{f,1}$. It was already shown that the action of $G_{f,1}$ on M' is trivial; thus

$$\mathbf{Mod}(\hat{S}_1)M'_{G_{f,1}} \cong \mathbf{Mod}(\hat{S}_1)\mathbf{Mod}(\hat{S}_1)_1 \otimes 1\{\cdot\}_{G_{f,1}}.$$

Then the decomposition in Theorem 6.16 takes the form

$$\mathbf{Mod}(S^2, A, \mathcal{C})(\mathbb{Z}^{\mathcal{C}} \times \mathbf{Mod}(\hat{S}_0) \times M' \times \mathbf{Mod}(\hat{S}_2))_{\mathbf{eMod}(S^2, A, \mathcal{C}) \times G_f} \otimes \mathbf{Mod}(S^2, A, \mathcal{C}).$$

8.3.3. Computation of the centralizer. Since the spectrum radius of \mathcal{C} is greater than 1, the canonical obstruction of f contains \mathcal{C} ; but since \mathcal{C} separates f into rational and finite order maps, \mathcal{C} is the canonical obstruction [29]. (Note that every curve in S_0 or S_2 is a Levy cycle; but such curves are not part of the canonical obstruction.) Therefore, $Z(f)$ fixes \mathcal{C} and we may write

$$Z(f) \leq \mathbf{Mod}(S^2, A, \mathcal{C}) \cong \mathbf{Mod}(\hat{S}_0) \times \mathbf{Mod}(\hat{S}_1) \times \mathbf{Mod}(\hat{S}_2) \times \mathbb{Z}^{\mathcal{C}}.$$

Furthermore, the projection of $Z(f)$ into $\mathbf{Mod}(\hat{S}_1)$ is trivial because the restriction of f to S_1 is a rational self-map. Therefore, $Z(f)$ is a subgroup of $\mathbf{Mod}(S_0) \times \mathbf{Mod}(S_2) \times \mathbb{Z}^{\mathcal{C}}$.

Consider an element $(n_0, n_1, n_2, v) \in \mathbf{Mod}(\hat{S}_0) \times \mathbf{Mod}(\hat{S}_1) \times \mathbf{Mod}(\hat{S}_2) \times \mathbb{Z}^{\mathcal{C}}$. Then $(n_0, n_1, n_2, v) \in Z(f)$ if and only if $(n_0, n_1, n_2, v) \cdot f = f \cdot (n_0, n_1, n_2, v)$; namely, if $n_1 = 1$ and $v = \theta(n_0, n_1, n_2) + T_{f, \mathcal{C}}(v)$. Writing $v = (v_s, v_t)$, we obtain the equations $\theta_{2,s}(n_2) = \theta_{2,t}(n_2) = -v_t$. Therefore, $Z(f)$ is isomorphic to

$$\mathbf{Mod}(\hat{S}_0) \times \mathbb{Z}^{\{s\}} \times \{n \in \mathbf{Mod}(\hat{S}_2) \mid \theta_{2,s}(n_2) = \theta_{2,t}(n_2)\}.$$

It is easy to see that the last factor is infinitely generated, since it is the kernel of an epimorphism from $\mathbf{Mod}(\hat{S}_2)$ to \mathbb{Z} . We compute it explicitly below in one case.

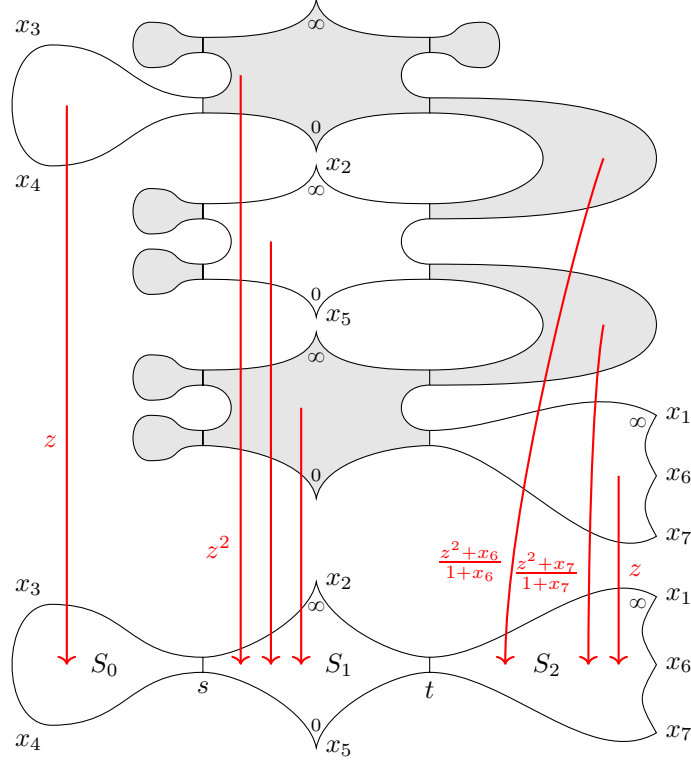


FIGURE 5. A Thurston map with infinitely generated centralizer

8.3.4. *The case $\ell = 1$ and $\#A = 7$.* These seem to produce the Thurston map of smallest degree and size of critical set whose centralizer is infinitely generated. The map f has degree 6 and has 7 marked points, see Figure 5. They are labeled as $A_0 = \{x_3, x_4\}$, $A_1 = \{x_2, x_5\}$ and $A_2 = \{x_1, x_6, x_7\}$.

To compute the centralizer of f , we write down a presentation of $B(f)$, and compute some relations in its mapping class biset. We set

$$G = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \mid x_1 x_2 x_3 x_4 x_5 x_6 x_7 \rangle,$$

write $s = x_3 x_4$ and $t = x_2 x_3 x_4 x_5$, and in a basis $\{\ell_1, \dots, \ell_7\}$ we compute the presentation

$$\begin{aligned}
 x_1 &= \langle\langle 1, s, s^{-1}, t, t^{-1}, x_1 \rangle\rangle(2, 3)(4, 5), \\
 x_2 &= \langle\langle 1, 1, s^{-1}, x_2 s, t^{-1}, t \rangle\rangle(1, 2)(3, 4)(5, 6), \\
 x_3 &= \langle\langle x_3, 1, 1, 1, 1, 1 \rangle\rangle, \\
 x_4 &= \langle\langle x_4, 1, 1, 1, 1, 1 \rangle\rangle, \\
 x_5 &= \langle\langle 1, 1, x_5, 1, 1, 1 \rangle\rangle(1, 2)(3, 4)(5, 6), \\
 x_6 &= \langle\langle 1, 1, 1, 1, 1, x_6 \rangle\rangle(2, 3), \\
 x_7 &= \langle\langle 1, 1, 1, 1, 1, x_7 \rangle\rangle(4, 5),
 \end{aligned}
 \tag{57}$$

giving $s = \langle\langle s, 1, 1, 1, 1, 1 \rangle\rangle$ and $t = \langle\langle 1, s, s^{-1}, t, t^{-1}, t \rangle\rangle$. We write $\sigma, \tau, \alpha, \beta$ for Dehn twists about s, t, x_1x_6 and x_6x_7 respectively; their actions on G are given respectively by

$$\begin{aligned}\sigma : \quad & x_3 \mapsto x_3^s, \quad x_4 \mapsto x_4^s, \\ \tau : \quad & x_2 \mapsto x_2^t, \quad x_3 \mapsto x_3^t, \quad x_4 \mapsto x_4^t, \quad x_5 \mapsto x_5^t, \\ \alpha : \quad & x_1 \mapsto x_1^{tx_6t^{-1}}, \quad x_6 \mapsto x_6^{t^{-1}x_1tx_6}, \\ \beta : \quad & x_6 \mapsto x_6^{x_6x_7}, \quad x_7 \mapsto x_7^{x_6x_7},\end{aligned}$$

all other generators being fixed. Naturally $[\sigma, \alpha] = [\tau, \alpha] = [\sigma, \beta] = [\tau, \beta] = 1$ while $\langle\alpha, \beta\rangle$ is a free group of rank 2. We then compute

$$\begin{aligned}B(f) \cdot \sigma &\cong \sigma \cdot B(f), & B(f) \cdot \tau &\cong \sigma^2 \tau^3 \cdot B(f), \\ B(f) \cdot \alpha &\cong \alpha \cdot \sigma^2 B(f), & B(f) \cdot \beta &\cong \beta \cdot B(f).\end{aligned}$$

For the second equality, the recursion of $\sigma^{-2}\tau^{-3} \cdot B(f) \cdot \tau$ in basis $\{s^2t^3\ell_1, st^3\ell_2, st^3\ell_3, t^2\ell_4, t^2\ell_5, \ell_6\}$ coincides with (57), while for the third equality, the recursion of $\sigma^{-2}\alpha^{-1} \cdot B(f) \cdot \alpha$ in basis $\{s^2\ell_1, s^2\ell_2, \ell_3, \dots, \ell_6\}$ coincides with (57).

Consider the homomorphism $\phi: \langle\alpha, \beta\rangle \rightarrow \mathbb{Z}$ which counts the total exponent in α of a word; it is the quotient by the normal closure of β . Then, for $w \in \langle\alpha, \beta\rangle$, the element $w\sigma^m\tau^n$ belongs to the centralizer of f if and only if $(m, n) = (m + 2n + \phi(w), 3n)$, if and only if $n = 0$ and $w \in \ker(\phi)$. Therefore,

$$Z(f) = \langle\sigma\rangle \times \ker(\phi) = \langle\sigma\rangle \times \langle\beta, \beta^\alpha, \beta^{\alpha\beta}, \dots\rangle \cong \mathbb{Z} \times F_\infty.$$

These calculations were checked using the computer algebra program GAP [16], and its package IMG, specially designed to manipulate wreath recursions. The commands issued were:

```
gap> m := NewSphereMachine("x1=<,x3*x4,x4^-1*x3^-1,x2*x3*x4*x5,x5^-1*x4^-1*x3^-1*x2^-1,x1>(2,3)(4,5)",
"x2=<,,x4^-1*x3^-1,x2*x3*x4,x5^-1*x4^-1*x3^-1*x2^-1,x2*x3*x4*x5>(1,2)(3,4)(5,6)",
"x3=<x3,,,,>","x4=<x4,,,,>","x5=<,,x5,,,,>(1,2)(3,4)(5,6)",
"x6=<,,,x6>(2,3)","x7=<,,,x7>(4,5)","x1*x2*x3*x4*x5*x6*x7");;
gap> g := StateSet(m);;
gap> AssignGeneratorVariables(g);; s := x3*x4; t := x2*s*x5;
gap> sigma := GroupHomomorphismByImages(g,[x1,x2,x3^s,x4^s,x5,x6,x7]);;
gap> tau := GroupHomomorphismByImages(g,[x1,x2^t,x3^t,x4^t,x5^t,x6,x7]);;
gap> alpha := GroupHomomorphismByImages(g,[x1^(t*x6/t),x2,x3,x4,x5,x6^(x1^t*x6),x7]);;
gap> beta := GroupHomomorphismByImages(g,[x1,x2,x3,x4,x5,x6^x7,x7^(x6*x7)]);;

gap> sigma*m = m*sigma;
true
gap> ChangeFRMachineBasis(tau*m/sigma^2/tau^3,[t^3*s^2,t^3*s,t^3*s,t^2,t^2,t^0]) = m;
true
gap> beta*m = m*beta;
true
gap> ChangeFRMachineBasis(alpha*m/alpha/sigma^2,[s^2,s^2,s^0,s^0,s^0,t^0]) = m;
true
```

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